

# RELATIONS IN THE 24-TH HOMOTOPY GROUPS OF SPHERES

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## Abstract

The main purpose of this note is to give a proof of the fact that the Toda brackets  $\langle \bar{\nu}, \sigma, \bar{\nu} \rangle$  and  $\langle \nu, \eta, \bar{\sigma} \rangle$  are not trivial. This is an affirmative answer of the second author's Conjecture (Determination of the  $P$ -image by Toda brackets, Geometry and Topology Monographs **13**(2008), 355–383). The second purpose is to show the relation  $\bar{\nu}_7 \omega_{15} = \nu_7 \sigma_{10} \kappa_{17}$  in  $\pi_{31}^7$ .

## 1 Introduction

Throughout this note, we work in the 2-primary components of homotopy groups of spheres. In the stable group, let  $\iota \in \pi_0^s$ ,  $\eta \in \pi_1^s$ ,  $\nu \in \pi_3^s$ ,  $\sigma \in \pi_7^s$ ,  $\bar{\nu}$ ,  $\varepsilon \in \pi_8^s$ ,  $\mu \in \pi_9^s$ ,  $\zeta \in \pi_{11}^s$ ,  $\kappa \in \pi_{14}^s$ ,  $\rho \in \pi_{15}^s$ ,  $\omega$ ,  $\eta^* \in \pi_{16}^s$ ,  $\bar{\mu} \in \pi_{17}^s$ ,  $\nu^*$ ,  $\xi \in \pi_{18}^s$ ,  $\bar{\zeta}$ ,  $\bar{\sigma} \in \pi_{19}^s$ ,  $\bar{\kappa} \in \pi_{20}^s$ ,  $\bar{\rho} \in \pi_{23}^s$ ,  $\delta \in \pi_{24}^s$ ,  $\mu_3 \in \pi_{25}^s$  be the generators [17, 9].

We know the following [9]:

$$\pi_{24}^s = \mathbf{Z}_2\{\bar{\mu}\sigma\} \oplus \mathbf{Z}_2\{\eta\eta^*\sigma\}.$$

The main purpose of this note is to give a proof of the fact that the Toda brackets  $\langle \bar{\nu}, \sigma, \bar{\nu} \rangle$  and  $\langle \nu, \eta, \bar{\sigma} \rangle$  are not trivial.

**Theorem 1.1**  $\eta_9 \sigma_{10} \eta_{17}^* = \{\varepsilon_9, \sigma_{17}, \eta_{24} \sigma_{25}\}_4$ ,  $\{\bar{\nu}_{20}, \sigma_{28}, \bar{\nu}_{35}\} = \{\nu_{20}, \eta_{23}, \bar{\sigma}_{24}\} = \eta_{20} \sigma_{21} \eta_{28}^* = \eta_{20} \eta_{21}^* \sigma_{37}$  and  $\langle \bar{\nu}, \sigma, \bar{\nu} \rangle = \langle \nu, \eta, \bar{\sigma} \rangle = \eta \eta^* \sigma$ .

This result gives an affirmative answer to [12, Conjecture 4.8]. In the proof of Theorem 1.1, our method is to inspect relations in homotopy groups of spheres through those in homotopy groups of rotation groups.

We know the following [17, Theorem 12.22]:

$$\pi_{31}^{13} = \mathbf{Z}_8\{\xi_{13}\} \oplus \mathbf{Z}_8\{\lambda\} \oplus \mathbf{Z}_2\{\eta_{13} \bar{\mu}_{14}\}.$$

Let  $P : \pi_{32}^{13} \rightarrow \pi_{30}^6$  be the  $P$ -homomorphism ( $P = \Delta$  in [17]). We need

**Lemma 1.2**  $H(P\xi_{13}) \equiv \xi' \pmod{2\lambda', 2\xi'}$  and  $H(P\lambda) \equiv \lambda' \pmod{2\lambda', 2\xi'}$ .

Notice that Lemma 1.2 improves [9, (3.3)].

Oda [13, Proposition 2.6 (5)] obtained the following relation in  $\pi_{31}^7$ :

$$\bar{\nu}_7 \omega_{15} \equiv 0 \pmod{\nu_7 \sigma_{10} \kappa_{17}, \bar{\zeta}_7'},$$

where  $\bar{\zeta}_7' = \sigma' \varepsilon_{14} \mu_{22}$  [9, (5.10)]. The second purpose of this note is to show

**Theorem 1.3**  $\bar{\nu}_6 \omega_{14} \equiv \nu_6 \sigma_9 \kappa_{16} + P(\xi_{13} + \lambda) \circ \eta_{29} \pmod{4\bar{\zeta}_6'}$  and  $\bar{\nu}_7 \omega_{15} = \nu_7 \sigma_{10} \kappa_{17}$ .

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## 2 Recollection of some relations in homotopy groups of spheres

We use the result, the notation of [17] and the properties of Toda brackets freely. We know

$$(2.1) \quad [\iota_2, \iota_2] = 2\eta_2,$$

$$(2.2) \quad \pm [\iota_4, \iota_4] = 2\nu_4 - E\nu',$$

$$(2.3) \quad \eta_4 \nu_5 = (E\nu')\eta_7 = [\iota_4, \eta_4], \quad \eta_5 \nu_6 = 0,$$

$$(2.4) \quad \nu_5 \eta_8 = [\iota_5, \iota_5],$$

$$(2.5) \quad 2\sigma_8 - E\sigma' = \pm[\iota_8, \iota_8]$$

and [17, Lemma 6.3]

$$(2.6) \quad \eta_5 \bar{\nu}_6 = \nu_5^3 \text{ and } \bar{\nu}_6 \eta_{14} = \nu_6^3.$$

We also know that

$$(2.7) \quad \eta_7 \sigma_8 = \bar{\nu}_7 + \varepsilon_7 + \sigma' \eta_{14} \quad [17, (7.4)],$$

$$(2.8) \quad \eta_9 \sigma_{10} = \bar{\nu}_9 + \varepsilon_9 \quad [17, \text{Lemma 6.4}],$$

and  $\eta_9 \sigma_{10} + \sigma_9 \eta_{16} = [\iota_9, \iota_9]$  [17, (7.1)]. We recall from [17, (7.19)] the relation

$$(2.9) \quad \sigma' \nu_{14} = x \nu_7 \sigma_{10} \quad (x : \text{odd}).$$

By (2.9) and [17, Lemma 5.14, (7.21)], we have

$$(2.10) \quad \nu_{10} \sigma_{13} = 2\sigma_{10} \nu_{17} = [\iota_{10}, \eta_{10}]$$

and

$$(2.11) \quad \sigma_{11}\nu_{18} = [\iota_{11}, \iota_{11}].$$

By [17, (7.18), (7.22)], we have relations

$$\nu_6\varepsilon_9 = (2\bar{\nu}_6)\nu_{14} = [\iota_6, \nu_6^2]$$

and

$$(2.12) \quad \bar{\nu}_9\nu_{17} = [\iota_9, \nu_9].$$

We recall the relations:

$$4\zeta_5 = \eta_5^2\mu_7, \text{ and } 4\bar{\zeta}_5 = \eta_5^2\bar{\mu}_7 \quad [17, (7.14), \text{Lemma 12.4}],$$

$$(2.13) \quad \eta_4\zeta_5 \equiv (E\nu')\mu_7 \pmod{(E\nu')\eta_7\varepsilon_8} \quad [15, \text{Proposition 2.2(5)}],$$

$$(2.14) \quad \varepsilon_3\sigma_{11} = 0 \text{ and } \bar{\nu}_6\sigma_{14} = 0. \quad [17, \text{Lemma 10.7}],$$

$$[\iota_{17}, \iota_{17}] \equiv \eta_{17}^* + \omega_{17} \pmod{\sigma_{17}\mu_{24}} \quad [17, \text{Proposition 12.20.ii)],}$$

$$(2.15) \quad \nu_7\kappa_{10} = \kappa_7\nu_{21} \quad [15, \text{Proposition 2.13(2)}],$$

$$(2.16) \quad [\iota_{19}, \iota_{19}] = \nu_{19}^* + \xi_{19} \quad [17, \text{Corollary 12.25}],$$

$$(2.17) \quad \nu_9\xi_{12} = \sigma_9^3 \text{ and } \xi_{12}\nu_{30} = \sigma_{12}^3 \quad [13, \text{II-Proposition 2.1(2)}],$$

$$(2.18) \quad \eta_6\kappa_7 = \bar{\varepsilon}_6, \quad \kappa_9\eta_{23} = \bar{\varepsilon}_9 \quad [17, (10.23)],$$

$$(2.19) \quad \nu_5\sigma_8\nu_{15}^2 = \eta_5\bar{\varepsilon}_6, \quad \varepsilon_3^2 = \varepsilon_3\bar{\nu}_{11} = \eta_3\bar{\varepsilon}_4 = \bar{\varepsilon}_3\eta_{18} \quad [17, \text{Lemma 12.10}],$$

$$\mu_3\varepsilon_{12} \equiv \eta_3\mu_4\sigma_{13} \pmod{2\bar{\varepsilon}'} \quad [15, \text{Proposition 2.13(7)}],$$

and

$$(2.20) \quad \mu_5\bar{\nu}_{14} = 0 \text{ and } \bar{\nu}_6\mu_{14} = 0 \quad [15, \text{Proposition 2.13(8)}].$$

By [17, Lemma 9.2] and the relation  $8\sigma_{14}^2 = 0$ , we have

$$(2.21) \quad \nu_7\zeta_{10} = (E^2\sigma''')\sigma_{14} \text{ and } \nu_{14}\zeta_{17} = 0.$$

By [15, Proposition 2.2(6)] and [17, (7.25)], we have

$$(2.22) \quad \zeta_6\eta_{17} = \nu_6\mu_9 = 8([\iota_6, \iota_6]\sigma_{11}).$$

By the relations (2.8) and  $\bar{\nu}_9\sigma_{17} = \varepsilon_9\sigma_{17} = 0$  [17, Lemma 10.7], we have

$$(2.23) \quad \eta_9\sigma_{10}^2 = (\bar{\nu}_9 + \varepsilon_9)\sigma_{17} = \bar{\nu}_9\sigma_{17} + \varepsilon_9\sigma_{17} = 0.$$

We recall from [17, (10.18), (10.20)] that  $\sigma_9(\bar{\nu}_{16} + \varepsilon_{16}) = [\iota_9, \sigma_9]$ ,  $\sigma_{10}\bar{\nu}_{17} = \sigma_{10}\varepsilon_{17} = [\iota_{10}, \nu_{10}^2]$  and

$$(2.24) \quad \sigma_{11}\bar{\nu}_{18} = \sigma_{11}\varepsilon_{18} = 0.$$

By (2.19) and (2.11), we have

$$(2.25) \quad \eta_9\bar{\varepsilon}_{10} = 0.$$

We recall the elements  $\lambda'$  and  $\xi'$  in  $\pi_{29}^{11}$  from [17, Lemma 12.19] and [14, Proposition 4(3)]:

$$(2.26) \quad E^2\lambda' = 2\lambda, \quad H(\lambda') = \varepsilon_{21}$$

and

$$(2.27) \quad E^2\xi' = 2\xi_{13}, \quad H(\xi') = \bar{\nu}_{21} + \varepsilon_{21} = \eta_{21}\sigma_{22}.$$

By (2.26), (2.27) and the proof of [15, Proposition 2.20(6)], we obtain

$$(2.28) \quad \nu_{11}\omega_{14} = (\lambda' + \xi')\eta_{29} \text{ and } \nu_{13}\omega_{16} = (2\lambda + 2\xi_{13})\eta_{31} = 0.$$

We also recall [17, (12.23)]

$$(2.29) \quad 4\sigma_{10}\zeta_{17} = 2\sigma_{11}\zeta_{18} = \sigma_{13}\zeta_{20} = \zeta_{13}\sigma_{24} = 0.$$

By (2.29), [17, Theorem 12.8, (12.25), p. 166] and using the EHP sequence, we obtain

$$(2.30) \quad 4\sigma_9\zeta_{16} = [\iota_9, \eta_9\mu_{10}], \quad 2\sigma_{10}\zeta_{17} = [\iota_{10}, \mu_{10}], \quad \sigma_{12}\zeta_{19} = 8[\iota_{12}, \sigma_{12}].$$

By [15, Proposition 2.17(7)] and (2.29), we have

$$(2.31) \quad \nu_{10}\rho_{13} \equiv 0 \pmod{2\sigma_{10}\zeta_{17}} \text{ and } \nu_{11}\rho_{14} = 0.$$

By the relations  $\bar{\zeta}_5\eta_{24} \equiv \zeta_5\mu_{16} \equiv \nu_5\bar{\mu}_8 \pmod{\nu_5\eta_8\mu_9\sigma_{18}}$  [13, II-Proposition 2.2(1);(2)] and  $\nu_6\bar{\mu}_9 = 16(P\rho_{13})$  [10, (16.6)], we have

$$\bar{\zeta}_6\eta_{25} = \zeta_6\mu_{17} = \nu_6\bar{\mu}_9 = 16(P\rho_{13}).$$

We recall from [17, Theorem 12.6, (12.4)] that :

$$\pi_{23}^7 = \{\sigma'\mu_{14}, \sigma'\eta_{14}\varepsilon_{15}, \mu_7\sigma_{16}, \eta_7\bar{\varepsilon}_8\} \quad (E\zeta' = \sigma'\eta_{14}\varepsilon_{15}).$$

By the relations  $E^2\sigma' = 2\sigma_9$  (2.5),  $2\eta_n = 0$  (2.1) and  $2\mu_n = 0$  for  $n \geq 3$  [17, Lemma 6.5], we have  $E^\infty(\sigma'\mu_{14}) = 0$  and  $E^\infty(\sigma'\eta_{14}\varepsilon_{15}) = 0$ . We know  $E^\infty(\eta_7\bar{\varepsilon}_8) = 0$  (2.25) and  $E^\infty(\mu_7\sigma_{16}) \neq 0$  [17, Theorem 12.16]. Hence we have

$$E^\infty\pi_{23}^7 = \{\sigma\mu\}.$$

We observe that  $\langle \zeta, \eta, \nu \rangle \circ \eta = \zeta \circ \langle \eta, \nu, \eta \rangle = \zeta \nu^2 = \zeta \nu \circ \nu = 0$  (2.21). Since  $\pi_{16}^s = \{\sigma\mu, \eta^*\} \cong (\mathbf{Z}_2)^2$  and  $(\eta)^* : \pi_{16}^s \rightarrow \pi_{17}^s$  is a monomorphism by [17, Theorem 12.16, 12.17], we have  $\langle \zeta, \eta, \nu \rangle = 0$ . Hence we obtain the relation

$$(2.32) \quad \mu_7 \sigma_{16} \notin \{\zeta_7, \eta_{18}, \nu_{19}\}.$$

Denote by  $\text{Ind}\{ , , \}_k$  the indeterminacy of the Toda bracket. We show

**Lemma 2.1**  $\zeta_5 \bar{\nu}_{16} = 0$  and  $\zeta_5 \varepsilon_{16} = \zeta_5 \eta_{16} \sigma_{17} = \nu_5 \mu_8 \sigma_{17}$ .

**Proof.** Since  $\bar{\nu}_{16} = \{\nu_{16}, \eta_{19}, \nu_{20}\}_1$  [17, Lemma 6.2] and  $\pi_{24}^{19} = 0$  [17, Proposition 5.9], we have

$$\zeta_5 \bar{\nu}_{16} \in \zeta_5 \circ \{\nu_{16}, \eta_{19}, \nu_{20}\}_1 \subset \{\zeta_5 \nu_{16}, \eta_{19}, \nu_{20}\}_1 \mod \pi_{21}^5 \circ \nu_{21}.$$

By the fact that  $\pi_{21}^5 = \{\mu_5 \sigma_{14}, \eta_5 \bar{\varepsilon}_6\}$  [17, Theorem 12.6] and the relation  $\sigma_{12} \nu_{19} = 0$  [17, (7.20)], we obtain  $\pi_{21}^5 \circ \nu_{21} = \{\eta_5 \bar{\varepsilon}_6 \nu_{21}\}$ . By the relations (2.18), (2.15) and  $\eta_6 \nu_7 = 0$  (2.3), we have

$$(2.33) \quad \eta_5 \bar{\varepsilon}_6 \nu_{21} = \eta_5 \eta_6 \kappa_7 \nu_{21} = \eta_5 \eta_6 \nu_7 \kappa_{10} = 0.$$

So, we get that  $\text{Ind}\{\zeta_5 \nu_{16}, \eta_{19}, \nu_{20}\}_1 = \pi_{21}^5 \circ \nu_{21} = 0$ .

We recall the relation [15, Proposition 2.4(2)]:

$$\zeta_5 \nu_{16} \equiv \nu_5 \zeta_8 \mod \nu_5 \bar{\nu}_8 \nu_{16}.$$

By the relation  $\bar{\nu}_5^2 = 0$  [15, Proposition 2.8(2)], we have

$$\{\nu_5 \bar{\nu}_8 \nu_{16}, \eta_{19}, \nu_{20}\}_1 \supset \nu_5 \bar{\nu}_8 \circ \{\nu_{16}, \eta_{19}, \nu_{20}\}_1 \ni \nu_5 \bar{\nu}_8^2 = 0 \mod \pi_{21}^5 \circ \nu_{21} = 0.$$

Hence, we obtain

$$\{\zeta_5 \nu_{16}, \eta_{19}, \nu_{20}\}_1 = \{\nu_5 \zeta_8, \eta_{19}, \nu_{20}\}_1.$$

We observe that

$$\{\nu_5 \zeta_8, \eta_{19}, \nu_{20}\}_1 \supset \nu_5 \circ \{\zeta_8, \eta_{19}, \nu_{20}\}_1 \mod \pi_{21}^5 \circ \nu_{21} = 0.$$

By the fact that  $\pi_{21}^8 = \{\sigma_8 \nu_{15}^2, \nu_8 \sigma_{11} \nu_{18}\} \cong (\mathbf{Z}_2)^2$  [17, Theorem 7.7] and the relation (2.19), we obtain

$$E\{\zeta_7, \eta_{18}, \nu_{19}\} \subset \{\zeta_8, \eta_{19}, \nu_{20}\}_1 \mod \pi_{21}^8 \circ \nu_{21} = \{\sigma_8 \nu_{15}^3, \eta_8 \bar{\varepsilon}_9\}.$$

So, by (2.32), we have

$$\{\zeta_8, \eta_{19}, \nu_{20}\}_1 \subset \{\sigma_8 \nu_{15}^3, \eta_8 \bar{\varepsilon}_9, (E\sigma')\mu_{15}, (E\sigma')\eta_{15} \varepsilon_{16}\}.$$

By the relations (2.19) and (2.33), we have  $\nu_5 \sigma_8 \nu_{15}^3 = \eta_5 \bar{\varepsilon}_6 \nu_{21} = 0$ . Hence, by (2.9) and  $2\nu_5 \sigma_8 = \nu_5 (E\sigma')$  [17, (7.16)], we see that  $\nu_5 \circ \{\zeta_8, \eta_{19}, \nu_{20}\}_1 = 0$ . This leads to the first half.

From the fact that  $\bar{\nu}_{16} = \varepsilon_{16} + \eta_{16}\sigma_{17}$  and the first half, we obtain  $\zeta_5\varepsilon_{16} = \zeta_5\eta_{16}\sigma_{17}$ . The last equality is just [15, Proposition 2.20(11)]. This leads to the second half and completes the proof.  $\square$

We recall the relation [17, Lemma 12.13]:

$$(2.34) \quad 16\bar{\sigma}_6 = \nu_6\mu_9\sigma_{18}.$$

By the group structures of  $\pi_{k+19}^k$  for  $k = 6, 7$  and the arguments in [17, p. 151-3] including [17, Lemma 12.13, (12.17)], we obtain

$$(2.35) \quad 2\bar{\sigma}_6 = yP(\sigma_{13}^2) \ (y : \text{odd}).$$

By Lemma 2.1 and (2.35), we have

$$(2.36) \quad \zeta_6\varepsilon_{17} = \zeta_6\eta_{17}\sigma_{18} = 16\bar{\sigma}_6 \text{ and } \zeta_7\varepsilon_{18} = 0.$$

We show

**Lemma 2.2** (1)  $\langle \sigma, \nu, \sigma \rangle = \xi$ .

(2)  $\langle \sigma, \nu, \eta\sigma \rangle = \langle \sigma, \nu, \varepsilon \rangle = \langle \nu, \sigma, \varepsilon \rangle = \langle \nu, \varepsilon, \sigma \rangle = 0$ .

**Proof.** Firstly, we recall that  $\pi_{11}^S = \{\zeta\}$ ,  $\pi_{12}^S = 0$ ,  $\pi_{16}^S = \{\eta\rho, \omega\}$  and the relations  $\eta\nu = 0$  (2.3),  $\sigma\zeta = 0$  (2.29),  $\varepsilon\zeta = 0$  (2.36),  $\nu\omega = 0$  (2.28). The indeterminacies of all brackets are 0, because  $\sigma \circ \pi_{11}^s = \pi_{11}^s \circ \sigma = 0$  and

$$\sigma \circ \pi_{12}^s = \pi_{12}^s \circ \sigma = \pi_{11}^s \circ \eta\sigma = \pi_{11}^s \circ \varepsilon = \nu \circ \pi_{16}^s = 0$$

The relation (1) follows directly from the definition of  $\xi_{12}$  [17, p.153].

By the fact that  $\langle \sigma, \nu, \eta \rangle \subset \pi_{12}^S = 0$ , we have

$$\langle \sigma, \nu, \eta\sigma \rangle \supset \langle \sigma, \nu, \eta \rangle \circ \sigma = 0 \text{ mod } 0.$$

By the Jacobi identity of Toda brackets [17, (3.7)], the definition of  $\varepsilon$  [17, (6.1)], the relation  $\langle \nu, \eta, 2\nu \rangle \subset \pi_5^S = 0$  and  $\pi_{13}^S = 0$ , we obtain

$$\begin{aligned} \langle \sigma, \nu, \varepsilon \rangle &= \langle \sigma, \nu, \langle \eta, 2\nu, \nu^2 \rangle \rangle \equiv \langle \sigma, \langle \nu, \eta, 2\nu \rangle, \nu^2 \rangle + \langle \langle \sigma, \nu, \eta \rangle, 2\nu, \nu^2 \rangle \\ &= \langle \sigma, 0, \nu^2 \rangle + \langle 0, 2\nu, \nu^2 \rangle \\ &\ni 0 \text{ mod } \sigma \circ \pi_{12}^s + \pi_{13}^s \circ \nu^2 = 0. \end{aligned}$$

By the relations  $\langle \eta\sigma, \nu, \sigma \rangle = \langle \nu, \sigma, \bar{\nu} \rangle = \bar{\sigma}$  [13, I-Proposition 3.3 (4)], (2.8) and using [17, (3.9.i)], we have

$$\begin{aligned} \langle \nu, \sigma, \varepsilon \rangle &= \langle \nu, \sigma, \bar{\nu} + \eta\sigma \rangle \subset \langle \nu, \sigma, \bar{\nu} \rangle + \langle \nu, \sigma, \eta\sigma \rangle \\ &= \langle \nu, \sigma, \bar{\nu} \rangle + \langle \eta\sigma, \nu, \sigma \rangle = \{\bar{\sigma} + \bar{\sigma}\} = 0. \end{aligned}$$

By the relations  $\langle \sigma, \nu, \varepsilon \rangle = 0$ ,  $\langle \nu, \sigma, \varepsilon \rangle = 0$  and use of [17, (3.9.ii), (3.10)], we have

$$\begin{aligned} 0 &\in \langle \nu, \varepsilon, \sigma \rangle + \langle \varepsilon, \sigma, \nu \rangle + \langle \sigma, \nu, \varepsilon \rangle \\ &= \langle \nu, \varepsilon, \sigma \rangle + \langle \nu, \sigma, \varepsilon \rangle + \langle \sigma, \nu, \varepsilon \rangle \\ &= \langle \nu, \varepsilon, \sigma \rangle \text{ mod } 0. \end{aligned}$$

This leads the last equality of (2) and completes the proof.  $\square$

We show

**Corollary 2.3**  $\{\nu_{11}, \sigma_{14}, \varepsilon_{21}\} \subset \{\lambda'\eta_{29}, \xi'\eta_{29}\}$  and  $E^2\{\nu_{11}, \sigma_{14}, \varepsilon_{21}\} = 0$ .

**Proof.** We recall  $\pi_{30}^{11} = \{\lambda'\eta_{29}, \xi'\eta_{29}, \bar{\zeta}_{11}, \bar{\sigma}_{11}\}$  [17, Theorem 12.23]. By stabilizing this result, Lemma 2.2 (2) and using the fact that  $\{\bar{\zeta}_{11}, \bar{\sigma}_{11}\} \cong \pi_{19}^s$ , we obtain

$$\{\nu_{11}, \sigma_{14}, \varepsilon_{21}\} \subset \{\lambda'\eta_{29}, \xi'\eta_{29}\}.$$

This leads to the first half.

The second half is obtained from the first half, (2.26) and (2.27). This completes the proof.  $\square$

### 3 Proof of Lemma 1.2

We recall the relation [13, I-Proposition 3.1(1)]

$$(3.1) \quad \nu_{10}\lambda = \sigma_{10}\kappa_{17}.$$

By (2.15) and (3.1), we obtain

$$(3.2) \quad \nu_{10}\lambda\nu_{31} = \sigma_{10}\nu_{17}\kappa_{20}.$$

By [13, II-Proposition 2.1(2)],

$$(3.3) \quad \kappa_7\sigma_{21} = 0.$$

By [14, Proposition 1(4)],

$$(3.4) \quad 2\xi'' \equiv \sigma_{10}\zeta_{17}, \text{ mod } 2\sigma_{10}\zeta_{17}.$$

Next we recall the element  $\omega' \in \pi_{31}^{12}$  [17, Lemma 12.21, (12.27), p. 166]:

$$(3.5) \quad E^2\omega' = 2\omega_{14}\nu_{30} = [\iota_{14}, \nu_{14}^2] \text{ and } H(\omega') \equiv \varepsilon_{23} \text{ mod } \varepsilon_{23} + \bar{\nu}_{23}.$$

By [17, (7.30), p. 166], we have

$$(3.6) \quad \xi_{13}\eta_{31} = [\iota_{13}, \sigma_{13}] = (E\theta)\sigma_{25}.$$

By the relations  $\nu_{13}\eta_{16}^* \equiv E\omega' \text{ mod } \xi_{13}\eta_{31}$  [15, Proposition 2.20(8)], (3.5) and (3.6), we have

$$(3.7) \quad \nu_{14}\eta_{17}^* = [\iota_{14}, \nu_{14}^2].$$

We show the following lemma overlapping with [15, Lemma 2.18]:

**Lemma 3.1** (1)  $\{2\sigma_{11}, \nu_{18}, \sigma_{21}\} \equiv \xi' \text{ mod } 2\lambda', 2\xi'$ .

$$(2) \{2\sigma_{11}, \nu_{18}, \varepsilon_{21}\} = \lambda' \eta_{29}.$$

$$(3) \xi_{12} \eta_{30} \equiv \theta \sigma_{24} \pmod{[\iota_{12}, \eta_{12} \sigma_{13}]}.$$

$$(4) \{\sigma_{12}, \nu_{19}, \varepsilon_{22}\} \ni \omega' + a \xi_{12} \eta_{30} \pmod{(E\lambda') \eta_{30}, (E\xi') \eta_{30}} \text{ for } a \in \{0, 1\}.$$

**Proof.** Since  $2\sigma_{11} \circ \pi_{30}^{18} = 0$ , we have

$$\{2\sigma_{11}, \nu_{18}, \sigma_{21}\} = \{2\sigma_{11}, \nu_{18}, \sigma_{21}\}_1.$$

By using [17, Proposition 2.6] and (2.10), we obtain

$$H\{2\sigma_{11}, \nu_{18}, \sigma_{21}\}_1 = -P^{-1}(2\sigma_{10}\nu_{17}) \circ \sigma_{22} = \{\eta_{21}\sigma_{22}\}.$$

This means  $H\{2\sigma_{11}, \nu_{18}, \sigma_{21}\}_1 = H\xi'$  from (2.27) and hence,  $\{2\sigma_{11}, \nu_{18}, \sigma_{21}\}_1 \ni \xi' \pmod{E\pi_{28}^{10}} = E\{\lambda'', \xi'', \eta_{10}\mu_{11}\} = \{2\lambda', 2\xi', \eta_{11}\bar{\mu}_{12}\}$ . By stabilizing this relation, we obtain

$$\{2\sigma_{11}, \nu_{18}, \sigma_{21}\}_1 \ni \xi' \pmod{2\lambda', 2\xi'}.$$

This leads to (1).

By the fact that  $\pi_{29}^{17} = 0$ ,  $\pi_{30}^{18} = 0$ ,  $\pi_{22}^{11} = \{\zeta_{11}\}$  [17, Theorem 7.4 and 7.6] and (2.36), we have

$$\text{Ind}\{2\sigma_{11}, \nu_{18}, \varepsilon_{21}\}_1 = 2\sigma_{11} \circ E\pi_{29}^{17} + \pi_{22}^{11} \circ \varepsilon_{22} = 0.$$

By using [17, Proposition 2.6] and (2.10), we obtain

$$H\{2\sigma_{11}, \nu_{18}, \varepsilon_{21}\}_1 = -P^{-1}(2\sigma_{10}\nu_{17}) \circ \varepsilon_{22} = \{\eta_{21}\varepsilon_{22}\}.$$

Since  $H(\lambda' \eta_{29}) = \varepsilon_{21} \eta_{29} = \eta_{21} \varepsilon_{22}$  from (2.26), we have  $\{2\sigma_{11}, \nu_{18}, \varepsilon_{21}\}_1 \ni \lambda' \eta_{29} \pmod{E\pi_{29}^{10}} = \{\zeta_{11}, \bar{\sigma}_{11}\}$ . This leads to (2).

Next we show (3). We recall  $\pi_{23}^{12} = \{\zeta_{12}, [\iota_{12}, \iota_{12}]\}$ . By the definitions of  $\xi_{12}$  and  $\theta$  [17, p153, Lemma 7.5], we have

$$\begin{aligned} \xi_{12} \eta_{30} &\in \{\sigma_{12}, \nu_{19}, \sigma_{22}\}_1 \circ \eta_{30} \\ &\subset \{\sigma_{12}, \nu_{19}, \sigma_{22} \eta_{29}\}_1 \\ &= \{\sigma_{12}, \nu_{19}, \eta_{22} \sigma_{23}\}_1 \\ &\supset \{\sigma_{12}, \nu_{19}, \eta_{22}\}_1 \circ \sigma_{24} \\ &\ni \theta \sigma_{24} \pmod{\sigma_{12} \circ E\pi_{30}^{18} + \pi_{23}^{12} \circ \eta_{23} \sigma_{24}}. \end{aligned}$$

By [17, Lemma 7.4 and 7.6] and (2.36), we get that  $\sigma_{12} \circ E\pi_{30}^{18} = 0$  and  $\pi_{23}^{12} \circ \eta_{23} \sigma_{24} = \{\zeta_{12}, [\iota_{12}, \iota_{12}]\} \circ \eta_{23} \sigma_{24} = \{[\iota_{12}, \eta_{12} \sigma_{12}]\}$ . This leads to (3).

We have  $\pi_{23}^{12} \circ \varepsilon_{23} = \{[\iota_{12}, \varepsilon_{12}]\}$ . By the fact that  $H(\lambda') = \varepsilon_{21}$  (2.26) and the argument in [5, Lemma 4.3], we obtain

$$\text{Ind}\{\sigma_{12}, \nu_{19}, \varepsilon_{22}\}_1 = \text{Ind}\{\sigma_{12}, \nu_{19}, \varepsilon_{22}\} = \{(E\lambda') \eta_{30}\}.$$



By using [17, Proposition 2.6], we obtain

$$(3.8) \quad H\{\sigma_{12}, \nu_{19}, \varepsilon_{22}\} = -P^{-1}(\sigma_{11}\nu_{18}) \circ \varepsilon_{23} = \{\varepsilon_{23}\}.$$

By the relations  $H(\xi_{12}\eta_{30}) = H(\xi_{12})\eta_{30} = \sigma_{23}\eta_{30}$  [17, Lemma 12.14], (2.7) and (3.5), we have  $H(\omega' + a\xi_{12}\eta_{30}) = \varepsilon_{23}$  for  $a = 0$  or  $a = 1$ . Hence we have  $H(\alpha) = H(\omega' + a\xi_{12}\eta_{30})$  for any representative  $\alpha$  in  $\{\sigma_{12}, \nu_{19}, \varepsilon_{22}\}$  and by the fact that  $E\pi_{30}^{11} = \{(E\lambda')\eta_{30}, (E\xi')\eta_{30}, \bar{\zeta}_{12}, \bar{\sigma}_{12}\}$  [17, Theorem 12.23], we have

$$\alpha - (\omega' + a\xi_{12}\eta_{30}) \in \{(E\lambda')\eta_{30}, (E\xi')\eta_{30}, \bar{\zeta}_{12}, \bar{\sigma}_{12}\}.$$

By stabilizing this relation and using the relation  $\langle \sigma, \nu, \varepsilon \rangle = 0$  (Lemma 2.2), the fact that  $\{\bar{\zeta}_{12}, \bar{\sigma}_{12}\} \cong \pi_{19}^s$ , we obtain

$$\alpha - (\omega' + a\xi_{12}\eta_{30}) \in \{(E\lambda')\eta_{30}, (E\xi')\eta_{30}\}.$$

This leads to (4) and completes the proof.  $\square$

We recall [13, II-Proposition 2.1(6), III-Proposition 2.2(2)]:

$$(3.9) \quad \xi'\sigma_{29} \equiv \sigma_{11}\nu_{18}^* \pmod{2\sigma_{11}\nu_{18}^*}$$

and

$$(3.10) \quad \lambda\sigma_{31} = 0.$$

Since  $H(\lambda'\sigma_{29}) = \varepsilon_{21} \circ \sigma_{29} = 0$  from (2.26) and [17, Lemma 10.7],  $E\pi_{35}^{10} = \{\sigma_{11}\xi_{18}, \sigma_{11}\nu_{18}^*, \mu_{3,11}, \eta_{11}\bar{\mu}_{12}\sigma_{29}\} \cong (\mathbf{Z}_4)^2 \oplus (\mathbf{Z}_2)^2$ ,  $\pi_{25}^s = \{\mu_3, \eta\bar{\mu}\sigma\} \cong (\mathbf{Z}_2)^2$  [13, Theorem 1(a)] and  $E^2(\lambda'\sigma_{29}) = 2\lambda\sigma_{31} = 0$ , we have

$$(3.11) \quad \lambda'\sigma_{29} \in \{\sigma_{11}\xi_{18}, \sigma_{11}\nu_{18}^*\}.$$

By the fact that  $E\theta' = [\iota_{12}, \eta_{12}]$  [17, (7.30)] and (2.18),

$$(3.12) \quad [\iota_{12}, \bar{\varepsilon}_{12}] = (E\theta')\kappa_{24}.$$

By [13, Theorem 1(b), I-Proposition 3.5(6)] and Lemma 3.1(3),

$$(3.13) \quad \xi_{12}\eta_{30}\sigma_{31} = \theta\sigma_{24}^2 = \sigma_{12}\bar{\sigma}_{19} \quad \text{and} \quad [\iota_{13}, \sigma_{13}^2] = \sigma_{13}\bar{\sigma}_{20} \neq 0.$$

We show

**Lemma 3.2**  $\omega'\sigma_{31} = 0$ .

**Proof.** Using the EHP sequence and the relation

$$H(\omega'\sigma_{31}) = H(\omega')\sigma_{31} \equiv \varepsilon_{23}\sigma_{31} = 0 \pmod{(\varepsilon_{23} + \bar{\nu}_{23})\sigma_{31} = 0}$$

from (3.5) and (2.14), we have  $\omega'\sigma_{31} \in E\pi_{37}^{11}$ . By (3.5) and [17, (7.20)], we have  $E^2(\omega'\sigma_{31}) = 2\omega_{14}\nu_{30}\sigma_{33} = 0$ . Hence, [13, I-Theorem 1(b), (8.19)] implies that there exist  $b, c \in \{0, 1\}$  satisfying the equation

$$\omega'\sigma_{31} = 4bE\tau''' + c[\iota_{12}, \bar{\varepsilon}_{12}]$$

and

$$E(\omega' \sigma_{31}) = 4bE^2 \tau''' = 32bE\tau^{IV} = 0.$$

By the EHP sequence, we have

$$\omega' \sigma_{31} \in P\pi_{40}^{25} = \{[\iota_{12}, \iota_{12}] \circ \rho_{23}, [\iota_{12}, \bar{\varepsilon}_{12}]\}.$$

By [13, Theorem 1(b), (8.19)], the order of  $[\iota_{12}, \iota_{12}] \circ \rho_{23} = 32\tau^{IV}$  is 32. This and (3.12) imply

$$\omega' \sigma_{31} \in \{32\tau^{IV}, (E\theta')\kappa_{24}\}.$$

On the other hand, by Lemma 3.1(4), we obtain

$$(\omega' + a\xi_{12}\eta_{30})\sigma_{31} \in \{\sigma_{12}, \nu_{19}, \varepsilon_{22}\} \circ \sigma_{31} \pmod{(E\lambda')\eta_{30}\sigma_{31}, (E\xi')\eta_{30}\sigma_{31}}.$$

By [17, Proposition 1.4], we have

$$\{\sigma_{12}, \nu_{19}, \varepsilon_{22}\} \circ \sigma_{31} = -\sigma_{12} \circ \{\nu_{19}, \varepsilon_{22}, \sigma_{30}\}.$$

By the fact that  $\pi_{38}^{19} \cong \pi_{19}^s$  and by Lemma 2.2(2), we obtain  $\{\sigma_{12}, \nu_{19}, \varepsilon_{22}\} \circ \sigma_{31} = 0$ . By (3.11), (3.6), [15, Proposition 2.20(3)] and (3.9), we have

$$(E\lambda')\eta_{30}\sigma_{31} = (E\lambda')\sigma_{30}\eta_{37} \in \{\sigma_{12}\xi_{19}, \sigma_{12}\nu_{19}^*\} \circ \eta_{37} = 0$$

and

$$(E\xi')\eta_{30}\sigma_{31} = (E\xi')\sigma_{30}\eta_{37} \equiv \sigma_{12}\nu_{19}^*\eta_{37} = 0 \pmod{2\sigma_{12}\nu_{19}^*\eta_{37} = 0}.$$

Hence, we have  $(\omega' + a\xi_{12}\eta_{30})\sigma_{31} = 0$  and the relation (3.13) implies

$$\omega' \sigma_{31} = a\sigma_{12}\bar{\sigma}_{19}.$$

Finally,  $32\tau^{IV}$ ,  $(E\theta')\kappa_{24}$  and  $\sigma_{12}\bar{\sigma}_{19}$  are independent in  $\pi_{38}^{12}$  [13, I-Theorem 1(b)]. This completes the proof.  $\square$

By this lemma and its proof, we obtain

$$(3.14) \quad \omega' \in \{\sigma_{12}, \nu_{19}, \varepsilon_{22}\} \pmod{(E\lambda')\eta_{30}, (E\xi')\eta_{30}}.$$

By [17, Theorem 10.10], (2.31), (2.18) and [5, (3.6)], we have  $\nu_{13} \circ \pi_{31}^{16} = \nu_{13} \circ \{\rho_{16}, \bar{\varepsilon}_{16}, [\iota_{16}, \iota_{16}]\} = 0$ . By [17, Theorem 7.4] and (2.29), we have  $\pi_{24}^{13} \circ \sigma_{24} = \{\zeta_{13}\sigma_{24}\} = 0$ . Hence, we obtain

$$\text{Ind}\{\nu_{13}, 2\sigma_{16}, \sigma_{23}\} = \nu_{13} \circ \pi_{31}^{16} + \pi_{24}^{13} \circ \sigma_{24} = 0.$$

Then, by [13, I-Proposition 3.4(8)], we have

$$(3.15) \quad \{\nu_{13}, 2\sigma_{16}, \sigma_{23}\} = \{\nu_{13}, \sigma_{16}, 2\sigma_{23}\} = \xi_{13} + x(\lambda + 2\xi_{13})$$

for some odd integer  $x$ .

We show

**Lemma 3.3** (1)  $H(\omega') = \varepsilon_{23}$ .

(2)  $\lambda\eta_{31} = E\omega' = \{\sigma_{13}, \nu_{20}, \varepsilon_{23}\}_n$  ( $n \leq 13$ ).

(3)  $\nu_{13}\eta_{16}^* = E\omega' + \xi_{13}\eta_{31}$ .

**Proof.** (1) follows from (3.14) and (3.8).

By [17, Theorem 7.4 and 7.6] and (2.36), we have

$$\text{Ind}\{\sigma_{13}, \nu_{20}, \varepsilon_{23}\}_n = \sigma_{13}E^n\pi_{32-n}^{20-n} + \pi_{24}^{13} \circ \varepsilon_{24} = 0 \quad (n \leq 13).$$

By the relations (3.10),  $\lambda\eta_{31} \equiv E\omega' \pmod{\xi_{13}\eta_{31}}$  [15, Proposition 2.20(2)], Lemma 3.2 and (3.13), we have

$$0 = \lambda\sigma_{31}\eta_{38} = \lambda\eta_{31}\sigma_{32} \equiv (E\omega')\sigma_{32} = 0 \pmod{\xi_{13}\eta_{31}\sigma_{31}} = [\iota_{13}, \sigma_{13}^2] \neq 0.$$

This leads to the first equality of (2). The second of (2) follows from (3.14), (2.26) and (2.27).

By the definitions of  $\sigma_{16}^*$  and  $\eta_{16}^*$  [17, p. 153], we obtain

$$\begin{aligned} \sigma_{16}^*\eta_{38} &\in \{\sigma_{16}, 2\sigma_{23}, \sigma_{30}\} \circ \eta_{38} \subset \{\sigma_{16}, 2\sigma_{23}, \sigma_{30}\eta_{37}\} \supset \{\sigma_{16}, 2\sigma_{23}, \eta_{30}\} \circ \sigma_{32} \\ &\ni \eta_{16}^*\sigma_{32} \pmod{\sigma_{16} \circ \pi_{39}^{23} + \pi_{31}^{16} \circ \eta_{31}\sigma_{32}}. \end{aligned}$$

Since  $\sigma_{16}\eta_{23}\rho_{24} = \sigma_{16}\mu_{23}\sigma_{32} = \rho_{16}\eta_{31}\sigma_{32} = \mu_{16}\sigma_{25}^2 = 0$  by [17, Proposition 12.20] and [9, (2.9)],  $\bar{\varepsilon}_{16}\eta_{31}\sigma_{32} = \eta_{16}\bar{\varepsilon}_{17}\sigma_{32} = 0$  by (2.25), we have  $\sigma_{16} \circ \pi_{39}^{23} = \{\sigma_{16}\eta_{23}^*\}$  and  $\pi_{31}^{16} \circ \eta_{31}\sigma_{32} = \{[\iota_{16}, \eta_{16}\sigma_{17}]\}$ . Hence, we have

$$\sigma_{16}^*\eta_{38} \equiv \eta_{16}^*\sigma_{32} \pmod{\sigma_{16}\eta_{23}^*, [\iota_{16}, \eta_{16}\sigma_{17}]}.$$

Since  $\sigma^* = 0$  and  $\eta^*\sigma = \sigma\eta^*$  in the stable range, we get that

$$\eta_{16}^*\sigma_{32} = \sigma_{16}\eta_{23}^* + \sigma_{16}^*\eta_{38} + a[\iota_{16}, \eta_{16}\sigma_{17}]$$

for  $a \in \{0, 1\}$ . By (3.15), (3.10) and (3.13), we see that

$$\begin{aligned} \nu_{13}\eta_{16}^*\sigma_{32} &= \nu_{13}(\sigma_{16}\eta_{23}^* + \sigma_{16}^*\eta_{38} + a[\iota_{16}, \eta_{16}\sigma_{17}]) = \nu_{13}\sigma_{16}^*\eta_{38} \\ &\in \nu_{13} \circ \{\sigma_{16}, 2\sigma_{23}, \sigma_{30}\} \circ \eta_{38} = \{\nu_{13}, \sigma_{16}, 2\sigma_{23}\} \circ \eta_{31}\sigma_{32} \\ &= \xi_{13}\eta_{31}\sigma_{32} + \lambda\sigma_{31}\eta_{38} = [\iota_{13}, \sigma_{13}^2]. \end{aligned}$$

Thus, by the relation  $\nu_{13}\eta_{16}^* \equiv E\omega' \pmod{\xi_{13}\eta_{31}}$  [15, Proposition 2.20(8)], we have the equation (3) and completes the proof.  $\square$

Here, we need the following property of the generalized  $P$ -homomorphism [18, 6].

**Lemma 3.4** Let  $k \geq 3$  and  $X, Y, Z$  and  $W$  be CW-complexes and  $\alpha \in [E^k Y, EX \wedge X]$ ,  $\beta \in [Z, Y]$  and  $\gamma \in [W, Z]$ . Suppose that the generalized  $P$ -homomorphisms  $P : [E^k A, EX \wedge X] \rightarrow [E^{k-2} A, X]$  for  $A = Y, EW$  and  $Y \cup_{\beta} CZ$  are well-defined and  $\alpha \circ E^k \beta = 0$  and  $\beta \circ \gamma = 0$ . Then the Toda bracket  $\{P(\alpha), E^{k-2}\beta, E^{k-2}\gamma\}_{k-2}$  is well-defined and

$$P\{\alpha, E^k \beta, E^k \gamma\}_k \subset \{P(\alpha), E^{k-2}\beta, E^{k-2}\gamma\}_{k-2}.$$

**Proof.** By Proposition 2.5 of [6], we have  $P(\alpha) \circ E^{k-2}\beta = P(\alpha \circ E^k\beta) = 0$ . Hence  $\{P(\alpha), E^{k-2}\beta, E^{k-2}\gamma\}_{k-2}$  is well-defined. We denote  $Y \cup_\beta CZ$  by  $C_\beta$  and the inclusion map  $Y \rightarrow C_\beta$  by  $i_\beta$ . It is well-known that  $C_{E^k\beta} = E^k C_\beta$ . By Proposition 1.7 of [17], any element of  $\{\alpha, E^k\beta, E^k\gamma\}_k$  is represented as  $(-1)^k \bar{\alpha} \circ E^k \tilde{\gamma}$ , where  $\bar{\alpha} \in [E^k C_\beta, EX \wedge X]$  is an extension of  $\alpha$  and  $\tilde{\gamma} \in [EW, E^k C_\beta]$  is a coextension of  $\gamma$ . By Proposition 2.5 of [6], we have  $P((-1)^k \bar{\alpha} \circ \Sigma^k \tilde{\gamma}) = (-1)^k P(\bar{\alpha}) \circ \Sigma^{k-2} \tilde{\gamma}$ . Since  $P(\bar{\alpha}) \circ \Sigma^{k-2} i_\beta = P(\bar{\alpha} \circ \Sigma^k i_\beta) = P(\alpha)$ , the element  $P(\bar{\alpha}) \in [\Sigma^{k-2} C_\beta, X]$  is an extension of  $P(\alpha)$ . Hence we obtain

$$P((-1)^k \bar{\alpha} \circ \Sigma^k \tilde{\gamma}) = (-1)^{k-2} P(\bar{\alpha}) \circ \Sigma^{k-2} \tilde{\gamma} \in \{P(\alpha), \Sigma^{k-2}\beta, \Sigma^{k-2}\gamma\}_{k-2}$$

and

$$P\{\alpha, \Sigma^k \beta, \Sigma^k \gamma\}_k \subset \{P(\alpha), \Sigma^{k-2}\beta, \Sigma^{k-2}\gamma\}_{k-2}.$$

□

Now we show Lemma 1.2.

**Lemma 3.5** (1)  $H(P\xi_{13}) \equiv \xi' \pmod{2\lambda', 2\xi'}$  and  $H(P\lambda) \equiv \lambda' \pmod{2\lambda', 2\xi'}$ .

(2)  $H(P(\xi_{13}\eta_{31})) = \xi'\eta_{29}$  and  $H(P(\lambda\eta_{31})) = \lambda'\eta_{29}$ .

**Proof.** (2) is a direct consequence of (1). By the definition of  $\xi_{12}$  and the fact that  $\sigma_{12} \circ E\pi_{29}^{18} = \sigma_{12} \circ E^2\pi_{28}^{17}$ , we know

$$\xi_{12} \in \{\sigma_{12}, \nu_{19}, \sigma_{22}\}_1 = \{\sigma_{12}, \nu_{19}, \sigma_{22}\}_2.$$

By Lemmas 3.1(1), 3.4, [17, Proposition 2.3] and  $HP\sigma_{13} = (H[\iota_6, \iota_6])\sigma_{11} = 2\sigma_{11}$ , we see that

$$P\xi_{13} \in P\{\sigma_{13}, \nu_{20}, \sigma_{23}\}_3 \subset \{P\sigma_{13}, \nu_{18}, \sigma_{21}\}_1$$

and

$$HP\xi_{13} \in \{2\sigma_{11}, \nu_{18}, \sigma_{21}\}_1 \equiv \xi' \pmod{2\lambda', 2\xi'}.$$

This leads to the first half of (1).

By the fact that  $\sigma_{12} \circ \pi_{31}^{19} = 2\sigma_{11} \circ \pi_{30}^{18} = 0$ , we have

$$\{\sigma_{12}, \nu_{19}, \varepsilon_{22}\} = \{\sigma_{12}, \nu_{19}, \varepsilon_{22}\}_n \quad (0 \leq n \leq 12)$$

and

$$\{2\sigma_{11}, \nu_{18}, \varepsilon_{21}\} = \{2\sigma_{11}, \nu_{18}, \varepsilon_{21}\}_n \quad (0 \leq n \leq 11).$$

So, by Lemmas 3.3(2) and 3.4, we obtain

$$P(E\omega') = P\{\sigma_{13}, \nu_{20}, \varepsilon_{23}\}_3 \subset \{P(\sigma_{13}), \nu_{18}, \varepsilon_{21}\}_1$$

and

$$HP(E\omega') \in H\{P(\sigma_{13}), \nu_{18}, \varepsilon_{21}\}_1 \subset \{2\sigma_{11}, \nu_{18}, \varepsilon_{21}\}_1 = \lambda'\eta_{29}.$$

By [9, (3.3)], we see that

$$H(P\lambda) \equiv \pm \lambda' \pmod{2\lambda', 2\xi', \eta_{11}\bar{\mu}_{12}}.$$

Hence, Lemma 3.3 implies

$$HP(\lambda\eta_{31}) = \lambda'\eta_{29}.$$

This and (1) lead to the second half of (1) and completes the proof.  $\square$

## 4 Proof of Theorem 1.1, I

We recall from [13, III-Proposition 2.6(5)] the relation

$$\bar{\nu}_7\omega_{15} \equiv 0 \pmod{\nu_7\sigma_{10}\kappa_{17}, \bar{\zeta}_7'}$$

and

$$(4.1) \quad \bar{\nu}_9\omega_{17} = 0.$$

We show

**Lemma 4.1** (1)  $\{\eta_{10}, \nu_{11}, \sigma_{14}\} = 2[\iota_{10}, \nu_{10}]$ .

(2)  $\{\eta_n, \nu_{n+1}, \sigma_{n+8}\} = 0$  ( $n \geq 11$ ).

**Proof.** First we show (2). From relations  $\eta_n\zeta_{n+1} = 0$  ( $n \geq 5$ ) (2.13) and  $\eta_{11} \circ [\iota_{12}, \iota_{12}] = [\iota_{11}, \eta_{11}^2] = 0$  [3, Theorem], it suffices to show that  $\{\eta_{11}, \nu_{12}, \sigma_{15}\} = 0$ . Since  $\eta_{11} \circ \pi_{23}^{12} + \pi_{16}^{11} \circ \sigma_{16} = \{\eta_{11}\zeta_{12}, \eta_{11} \circ [\iota_{12}, \iota_{12}]\} = 0$ , we have  $\text{Ind}\{\eta_{11}, \nu_{12}, \sigma_{15}\} = 0$ . From the fact that  $\{\eta_{10}, \nu_{11}, \sigma_{14}\} \subset \pi_{22}^{10} = \{[\iota_{10}, \nu_{10}]\}$ , we have  $\{\eta_{11}, \nu_{12}, \sigma_{15}\} \supset E\{\eta_{10}, \nu_{11}, \sigma_{14}\} = 0 \pmod{0}$ .

Next we show (1). Let  $\mathbf{HP}^2$  be the quaternionic projective plane and  $i_{\mathbf{H}} : S^4 \hookrightarrow \mathbf{HP}^2$  the inclusion. By the definition of the Toda bracket, there exist an extension  $\bar{\eta}'_9 \in [E^6\mathbf{HP}^2, S^9]$  of  $\eta_9$  and a coextension  $\bar{\sigma}_{14} \in \pi_{22}(E^7\mathbf{HP}^2)$  of  $\sigma_{14}$  such that  $\{\eta_{10}, \nu_{11}, \sigma_{14}\} = E\bar{\eta}'_9 \circ \bar{\sigma}_{14}$ . By the Blakers-Massey theorem, we have  $\pi_{22}(E^7\mathbf{HP}^2, S^{11}) \cong \pi_{22}(S^{15})$  and  $\pi_{23}(E^7\mathbf{HP}^2, S^{11}) \cong \pi_{23}(S^{15})$ . So, by the homotopy exact sequence of a pair  $(E^7\mathbf{HP}^2, S^{11})$ , we obtain

$$\pi_{22}(E^7\mathbf{HP}^2) = \{\bar{\sigma}_{14}, i\zeta_{11}\} \ (i = E^7i_{\mathbf{H}} : S^{11} \hookrightarrow E^7\mathbf{HP}^2).$$

We consider the EHP sequence

$$\pi_{22}(E^7\mathbf{HP}^2) \xrightarrow{H} \pi_{22}(E(E^6\mathbf{HP}^2 \wedge E^6\mathbf{HP}^2)) \xrightarrow{P} \pi_{20}(E^6\mathbf{HP}^2).$$

We have  $\pi_{21}(E^6\mathbf{HP}^2, S^{10}) \cong \pi_{21}(S^{14})$  and  $\partial\pi_{21}(E^6\mathbf{HP}^2, S^{10}) = \{\nu_{10}\sigma_{13}\}$ . So we obtain  $(E^6i_{\mathbf{H}})[\iota_{10}, \eta_{10}] = 0$ . We have  $\pi_{22}(E(E^6\mathbf{HP}^2 \wedge E^6\mathbf{HP}^2)) \cong \pi_{22}(S^{21})$  and  $P(\pi_{22}(E(E^6\mathbf{HP}^2 \wedge E^6\mathbf{HP}^2))) = \{(E^6i_{\mathbf{H}})[\iota_{10}, \eta_{10}]\} = 0$  (2.10). Hence we get that

$$H(\bar{\sigma}_{14}) = E(E^6i_{\mathbf{H}} \wedge E^6i_{\mathbf{H}})\eta_{21}.$$

Thus we see that

$$H(E\bar{\eta}'_9 \circ \tilde{\sigma}_{14}) = E(\bar{\eta}'_9 \wedge \bar{\eta}'_9) \circ H(\tilde{\sigma}_{14}) = E(\bar{\eta}'_9 \wedge \bar{\eta}'_9) \circ E(E^6 i_{\mathbf{H}} \wedge E^6 i_{\mathbf{H}}) \eta_{21} = \eta_{19}^3 = 4\nu_{19}.$$

This leads to (1) and completes the proof.  $\square$

We need relations

**Lemma 4.2**  $\{\nu_{13}, \sigma_{16}, \varepsilon_{23}\}_2 = 0$  and  $\{\nu_n, \sigma_{n+3}, \varepsilon_{n+10}\} = 0$  for  $n \geq 15$ .

**Proof.** By using  $\pi_{30}^{14} = \{\omega_{14}, \sigma_{14}\mu_{21}\}$ ,  $\pi_{k+16}^k = \{\omega_k, \sigma_k\mu_{k+7}\}$  for  $k \geq 18$ ,  $\pi_{\ell+11}^\ell = \{\zeta_\ell\}$  for  $\ell \geq 13$ , (2.28), (2.10) and (2.36), we have

$$\text{Ind}\{\nu_{13}, \sigma_{16}, \varepsilon_{23}\}_2 = \nu_{13} \circ E^2 \pi_{30}^{14} + \pi_{24}^{13} \circ \varepsilon_{24} = 0$$

and

$$\text{Ind}\{\nu_n, \sigma_{n+3}, \varepsilon_{n+10}\} = \nu_n \circ \pi_{n+19}^{n+3} + \pi_{n+11}^n \circ \varepsilon_{n+11} = 0$$

for  $n \geq 15$ . By Corollary 2.3, we have

$$0 = E^2\{\nu_{11}, \sigma_{14}, \varepsilon_{21}\} \subset \{\nu_{13}, \sigma_{16}, \varepsilon_{23}\}_2 \pmod{0}.$$

This leads to the first half. Moreover, we have

$$\begin{aligned} 0 &= E^{n-13}\{\nu_{13}, \sigma_{16}, \varepsilon_{23}\}_2 \subset E^{n-15}\{\nu_{15}, \sigma_{18}, \varepsilon_{25}\} \\ &\subset (-1)^{n-15}\{\nu_n, \sigma_{n+3}, \varepsilon_{n+10}\} \pmod{0}. \end{aligned}$$

This leads to the second half and completes the proof.  $\square$

We show

**Lemma 4.3**  $\{\bar{\nu}_{11}, \sigma_{19}, \varepsilon_{26}\} \ni 0 \pmod{\sigma_{11}\nu_{18}\kappa_{21} = [\iota_{11}, \kappa_{11}]}$  and  $E\{\bar{\nu}_{11}, \sigma_{19}, \varepsilon_{26}\} = \{\bar{\nu}_{12}, \sigma_{20}, \varepsilon_{27}\} = 0$ .

**Proof.** By [17, Theorem 12.16], we have

$$\text{Ind}\{\bar{\nu}_{11}, \sigma_{19}, \varepsilon_{26}\} = \bar{\nu}_{11} \circ \pi_{35}^{19} + \pi_{27}^{11} \circ \varepsilon_{27} = \bar{\nu}_{11} \circ \{\omega_{19}, \sigma_{19}\mu_{26}\} + \{\sigma_{11}\mu_{18}\} \circ \varepsilon_{27}.$$

By (2.14), (4.1) and the fact that  $\sigma_{11}\mu_{18}\varepsilon_{27} = \sigma_{11}\varepsilon_{18}\mu_{26} = 0$  from (2.24), the indeterminacy is trivial. Similarly, we obtain

$$\text{Ind}\{\bar{\nu}_{12}, \sigma_{20}, \varepsilon_{27}\} = 0.$$

Therefore, the second assertion follows directly from the first assertion.

By the relation  $\{\nu_n, \eta_{n+3}, \nu_{n+4}\} = \bar{\nu}_n$  for  $n \geq 7$  [17, Lemma 6.2], we have

$$\{\bar{\nu}_{11}, \sigma_{19}, \varepsilon_{26}\} = \{\{\nu_{11}, \eta_{14}, \nu_{15}\}, \sigma_{19}, \varepsilon_{26}\}.$$

By the Jacobi identity of Toda brackets and Lemmas 4.1 and 4.2, we have

$$\begin{aligned} \{\{\nu_{11}, \eta_{14}, \nu_{15}\}, \sigma_{19}, \varepsilon_{26}\} &\equiv \{\nu_{11}, \{\eta_{14}, \nu_{15}, \sigma_{18}\}, \varepsilon_{26}\} + \{\nu_{11}, \eta_{14}, \{\nu_{15}, \sigma_{18}, \varepsilon_{25}\}\} \\ &\ni 0 \pmod{\nu_{11} \circ \pi_{35}^{14} + \pi_{27}^{11} \circ \varepsilon_{27} = \nu_{11} \circ \pi_{35}^{14}}. \end{aligned}$$

By [8, Theorem A],  $\nu_{11}\eta_{14} = 0$  and  $\nu_{11}\sigma_{14} = 0$ , we have  $\nu_{11} \circ \pi_{35}^{14} = \{E(\nu_{10}\lambda\nu_{31})\}$ . Moreover, by the relations (2.11) and (3.2), we have  $E(\nu_{10}\lambda\nu_{31}) = \sigma_{11}\nu_{18}\kappa_{17} = [\iota_{11}, \kappa_{11}]$ . Thus, we have the first assertion. This completes the proof.  $\square$

Here we recall the definitions of  $\delta_3$ ,  $\bar{\sigma}'_6$  and  $\bar{\sigma}'_6$  [9, p. 13, 15]:

$$\delta_3 \in \{\varepsilon_3, \varepsilon_{11} + \bar{\nu}_{11}, \sigma_{19}\}_1,$$

$$\bar{\sigma}'_6 \in \{\bar{\nu}_6, \varepsilon_{14} + \bar{\nu}_{14}, \sigma_{22}\}_1,$$

$$\bar{\sigma}'_6 \in \{\nu_6, \eta_9, \bar{\sigma}_{10}\}_3.$$

The indeterminacy of  $\{\varepsilon_3, \varepsilon_{11} + \bar{\nu}_{11}, \sigma_{19}\}_1$  is  $\varepsilon_3 \circ E\pi_{26}^{10} + \pi_{20}^3 \circ \sigma_{20}$ . Since  $\varepsilon_3\sigma_{11} = 0$  (2.14), we have  $\varepsilon_3 \circ E\pi_{26}^{10} = 0$ . From the fact that  $\pi_{20}^3 = \{\bar{\varepsilon}', \bar{\mu}_3, \eta_3\mu_4\sigma_{13}\}$ ,  $\bar{\varepsilon}'\sigma_{20} = 0$  [13, I-Proposition 3.1(5)] and

$$(4.2) \quad \mu_3\sigma_{12}^2 = 0 \text{ [9, (2.9)]},$$

we obtain  $\pi_{20}^3 \circ \sigma_{20} = \{\bar{\mu}_3\sigma_{20}\}$ . Hence, we have

$$\text{Ind}\{\varepsilon_3, \varepsilon_{11} + \bar{\nu}_{11}, \sigma_{19}\}_1 = \{\bar{\mu}_3\sigma_{20}\}.$$

The indeterminacy of  $\{\bar{\nu}_6, \varepsilon_{14} + \bar{\nu}_{14}, \sigma_{22}\}_1$  is  $\bar{\nu}_6 \circ E\pi_{29}^{13} + \pi_{23}^6 \circ \sigma_{23}$ . By the fact that  $\pi_{29}^{13} = \{\sigma_{13}\mu_{20}\}$  [17, Theorem 12.16] and (2.14), we have  $\bar{\nu}_6 \circ E\pi_{29}^{13} = 0$ . By the fact that  $\pi_{23}^6 = \{P(E\theta), \nu_6\kappa_9, \bar{\mu}_6, \eta_6\mu_7\sigma_{16}\}$  [17, Theorem 12.7], (3.6) and (4.2), we obtain  $\pi_{23}^6 \circ \sigma_{23} = \{P(\xi_{13}\eta_{31})\}$ . Hence, we have

$$\text{Ind}\{\bar{\nu}_6, \varepsilon_{14} + \bar{\nu}_{14}, \sigma_{22}\}_1 = \{P(\xi_{13}\eta_{31})\}.$$

The indeterminacy of  $\{\nu_6, \eta_9, \bar{\sigma}_{10}\}_3$  is  $\nu_6 \circ E^3\pi_{24}^6 + \pi_{11}^6 \circ \bar{\sigma}_{11}$ . Since  $[\iota_6, \iota_6] \circ \bar{\sigma}_{11} = 0$  [9, (5.7)], we have  $\pi_{11}^6 \circ \bar{\sigma}_{11} = 0$ . From the fact that  $E^3\pi_{24}^6 = \{\sigma_9\zeta_{16}, \eta_9\bar{\mu}_{10}\}$ , we obtain  $\nu_6 \circ E^3\pi_{24}^6 = 0$ . Hence, we have

$$\text{Ind}\{\nu_6, \eta_9, \bar{\sigma}_{10}\}_3 = 0.$$

Since  $\varepsilon_n + \bar{\nu}_n = \sigma_n\eta_{n+7}$  for  $n \geq 10$  (2.8), and (2.14), we have

$$\{\varepsilon_3, \sigma_{11}, \eta_{18}\sigma_{19}\}_1 \subset \{\varepsilon_3, \varepsilon_{11} + \bar{\nu}_{11}, \sigma_{19}\}_1$$

and

$$\{\bar{\nu}_6, \sigma_{14}, \eta_{21}\sigma_{22}\}_1 \subset \{\bar{\nu}_6, \varepsilon_{14} + \bar{\nu}_{14}, \sigma_{22}\}_1.$$

We change the definitions of  $\delta_3$  and  $\bar{\sigma}'_6$  as follows:

$$(4.3) \quad \delta'_3 \in \{\varepsilon_3, \sigma_{11}, \eta_{18}\sigma_{19}\}_1,$$

$$(4.4) \quad \bar{\sigma}''_6 \in \{\bar{\nu}_6, \sigma_{14}, \eta_{21}\sigma_{22}\}_1.$$

By relations [13, I-Proposition 3.1.(2)] and (2.8), we have

$$\zeta'\varepsilon_{22} = \zeta'\bar{\nu}_{22} = 0 \text{ and so, } \zeta'\eta_{22}\sigma_{23} = 0.$$

By using [17, Theorem 12.6 and 12.16],  $\varepsilon_3\sigma_{11}\mu_{18} = 0$  (2.14),  $\mu_3\sigma_{12}\eta_{19}\sigma_{20} = \mu_3\eta_{12}\sigma_{13}^2 = 0$  (2.23),  $\eta_3\bar{\varepsilon}_4\eta_{19}\sigma_{20} = \eta_3\varepsilon_4\bar{\nu}_{12}\sigma_{20} = 0$  (2.19) and (2.14),  $\bar{\nu}_6\sigma_{14}\nu_{21} = 0$  (2.14),  $\mu_6\sigma_{15}\eta_{22}\sigma_{23} = \mu_6\eta_{15}\sigma_{16}^2 = 0$  (2.23) and  $\eta_6\bar{\varepsilon}_7\eta_{22}\sigma_{23} = \eta_6\bar{\varepsilon}_7\sigma_{22}\eta_{29} = 0$  (2.14), we have

$$\text{Ind}\{\varepsilon_3, \sigma_{11}, \eta_{18}\sigma_{19}\}_1 = \varepsilon_3 \circ \pi_{27}^{11} + \pi_{19}^3 \circ \eta_{19}\sigma_{20} = 0$$

and

$$\text{Ind}\{\bar{\nu}_6, \sigma_{14}, \eta_{21}\sigma_{22}\}_1 = \bar{\nu}_6 \circ E\pi_{29}^{13} + \pi_{22}^6 \circ \eta_{22}\sigma_{23} = \{\bar{\nu}_6\sigma_{14}\mu_{21}, \zeta_{11}\eta_{22}\sigma_{23}\} = 0.$$

It is easy to check that this changing gives no influences to the computations in [9].

By [17, Proposition 2.6] and (2.10), we have

$$H\{\nu_{11}, \sigma_{14}, \eta_{21}\sigma_{22}\}_1 = -P^{-1}(\nu_{10}\sigma_{13}) \circ \eta_{22}\sigma_{23} = \eta_{21}^2\sigma_{23}.$$

Hence, by the relations [9, (3.8)] and  $H(\xi') = \eta_{21}\sigma_{22}$  [17, Lemma 12.19], we take

$$\{\nu_{11}, \sigma_{14}, \eta_{21}\sigma_{22}\}_1 \ni \xi'\eta_{29} \mod E\pi_{29}^{10} = \{\bar{\sigma}_{11}, \bar{\zeta}_{11}\}.$$

Since  $\langle \nu, \sigma, \eta\sigma \rangle = \bar{\sigma}$  by [13, I-Proposition 3.3 (4)] and [17, (3.9).i)], we have

$$(4.5) \quad H(\bar{\sigma}_6'') = \bar{\sigma}_{11} + \xi'\eta_{29}.$$

We set

$$\delta'_n = E^{n-3}\delta'_3 \ (n \geq 3), \ \delta' = E^\infty\delta'_3$$

and

$$\bar{\sigma}_n'' = E^{n-6}\bar{\sigma}_6'' \ (n \geq 6), \ \bar{\sigma}'' = E^\infty\bar{\sigma}_6''.$$

Now we show

$$\textbf{Lemma 4.4} \quad \bar{\sigma}_{12}'' = \{\bar{\nu}_{12}, \sigma_{20}, \bar{\nu}_{27}\} = \{\nu_{12}, \eta_{15}, \bar{\sigma}_{16}\} = \bar{\sigma}_{12}'.$$

**Proof.** Notice that the indeterminacies of the brackets

$$\text{Ind}\{\bar{\nu}_{11}, \sigma_{19}, \bar{\nu}_{26}\} = \text{Ind}\{\bar{\nu}_{11}, \sigma_{19}, \eta_{26}\sigma_{27}\} = 0,$$

respectively, because  $\bar{\nu}_{11} \circ \pi_{35}^{19} = \{\bar{\nu}_{11}\omega_{19}, \bar{\nu}_{11}\sigma_{19}\mu_{26}\} = 0$  by (4.1) and (2.14),  $\pi_{27}^{11} \circ \bar{\nu}_{27} = \{\sigma_{11}\mu_{18}\bar{\nu}_{27}\} = 0$  by (2.20) and  $\pi_{27}^{11} \circ \eta_{27}\sigma_{28} = \{\mu_{11}\sigma_{20}\eta_{27}\sigma_{28}\} = 0$  by (2.23). So, by (2.8), Lemma 4.3 and (4.4), we obtain

$$\{\bar{\nu}_{11}, \sigma_{19}, \bar{\nu}_{26}\} = \{\bar{\nu}_{11}, \sigma_{19}, \varepsilon_{26}\} + \{\bar{\nu}_{11}, \sigma_{19}, \eta_{26}\sigma_{27}\} = a[\iota_{11}, \kappa_{11}] + \bar{\sigma}_{11}''$$

for  $a = 0$  or  $a = 1$ . Hence, we obtain

$$\bar{\sigma}_{12}'' \in E\{\bar{\nu}_{11}, \sigma_{19}, \bar{\nu}_{26}\} \subset \{\bar{\nu}_{12}, \sigma_{20}, \bar{\nu}_{27}\} \mod \bar{\nu}_{12} \circ \pi_{36}^{20} + \pi_{28}^{12} \circ \bar{\nu}_{28} = 0.$$



By the Jacobi identity of Toda brackets, [17, Lemma 6.2], Lemma 4.1.(2) and [13, I-Proposition 3.4(3)], we obtain

$$\begin{aligned}\bar{\sigma}_{12}'' &= \{\bar{\nu}_{12}, \sigma_{20}, \bar{\nu}_{27}\} = \{\{\nu_{12}, \eta_{15}, \nu_{16}\}, \sigma_{20}, \bar{\nu}_{27}\} \\ &\equiv \{\nu_{12}, \{\eta_{15}, \nu_{16}, \sigma_{19}\}, \bar{\nu}_{27}\} + \{\nu_{12}, \eta_{15}, \{\nu_{16}, \sigma_{19}, \bar{\nu}_{26}\}\} \\ &\equiv \{\nu_{12}, \eta_{15}, \bar{\sigma}_{16}\} \mod \nu_{12} \circ \pi_{36}^{15}.\end{aligned}$$

Since  $\pi_{36}^{15} = \{\eta_{15}\bar{\kappa}_{16}, \sigma_{15}^3, (E^2\lambda)\nu_{33}\} \cong (\mathbf{Z}_2)^3$  [8, Theorem A], (2.10) and (3.2), we have  $\nu_{12} \circ \pi_{36}^{15} = \{\nu_{12}(E^2\lambda)\nu_{33}\} = 0$ . This implies

$$\{\nu_{12}, \eta_{15}, \bar{\sigma}_{16}\} = \bar{\sigma}_{12}''.$$

and completes the proof.  $\square$

We show

**Lemma 4.5**  $\delta_3' \equiv \delta_3 \mod \bar{\mu}_3\sigma_{20}$ ,  $\bar{\sigma}_6'' \equiv \bar{\sigma}_6' \mod P(\xi_{13})\eta_{29}$  and  $\bar{\sigma}_6'' \equiv \bar{\sigma}_6' + P(\xi_{13})\eta_{29} \mod \nu_6\sigma_9\kappa_{16}$ .

**Proof.** By the fact that  $\pi_{20}^3 = \{\bar{\mu}_3, \eta_3\mu_4\sigma_{13}, \bar{\varepsilon}'\}$ ,  $\mu_3\sigma_{12}^2 = 0$  (4.2) and  $\bar{\varepsilon}'\sigma_{20} = 0$  [13, I-Proposition 3.1(5)], we get the first half.

We observe

$$\bar{\sigma}_6'' \in \{\bar{\nu}_6, \sigma_{14}, \eta_{21}\sigma_{22}\}_1 \subset \{\bar{\nu}_6, \varepsilon_{14} + \bar{\nu}_{14}, \sigma_{22}\}_1 \mod \bar{\nu}_6 \circ E\pi_{29}^{13} + \pi_{23}^6 \circ \sigma_{23}.$$

We know  $\pi_{23}^6 = \{P(E\theta), \nu_6\kappa_9, \bar{\mu}_6, \eta_6\mu_7\sigma_{16}\}$ ,  $\nu_6\kappa_9\sigma_{23} = 0$  (3.3) and  $P(E\theta)\sigma_{23} = P(\xi_{13}\eta_{31})$  (3.6). This leads to the second.

By the relations  $H(\bar{\sigma}_6') = \bar{\sigma}_{11}$  [9, (3.8)], (4.5) and Lemma 3.5(2), we have

$$H(\bar{\sigma}_6'') = H(\bar{\sigma}_6' + P(\xi_{13})\eta_{29})$$

and

$$\bar{\sigma}_6'' \equiv \bar{\sigma}_6' + P(\xi_{13})\eta_{29} \mod E\pi_{29}^5 = \{\delta_6, \bar{\mu}_6\sigma_{23}, \nu_6\sigma_9\kappa_{16}\}.$$

Hence, by the fact that  $\bar{\sigma}_{12}'' = \bar{\sigma}_{12}'$  (Lemma 4.4), we obtain the second half. This completes the proof.  $\square$

## 5 Proof of Theorem 1.1, II: Some relations in homotopy groups of the rotation group

Let  $SO(n)$  be the  $n$ -th rotation group,  $i_{k,n} : SO(k) \hookrightarrow SO(n)$  ( $k \leq n$ ) the inclusion,  $i_n = i_{n,n+1}$  and  $p_n : SO(n) \rightarrow SO(n)/SO(n-1) = S^{n-1}$  the projection. Let  $\Delta : \pi_k(S^n) \rightarrow \pi_{k-1}(SO(n))$  be the connecting map associated with the bundle  $p_{n+1} : SO(n+1) \rightarrow S^n$ . Suppose that there exist elements  $\alpha \in \pi_k(S^n)$  and  $\beta \in \pi_k(SO(n+1))$  satisfying the relation

$$p_{n+1*}\beta = \alpha.$$

Then,  $\beta$  is called a lift of  $\alpha$  and is written  $[\alpha]$ . For a lift  $[\alpha] \in \pi_k(SO(m))$ , we write  $[\alpha]_n = i_{m,n*}[\alpha] \in \pi_k(SO(n))$  for  $m \leq n$ .

Let  $J : \pi_k(SO(n)) \rightarrow \pi_{k+n}(S^n)$  be the  $J$ -homomorphism. Denote by  $R_k^n$  the 2-primary component of  $\pi_k(SO(n))$ . We use the exact sequence induced from the fibration  $p_n$ :

$$(\mathcal{R}_k^n) \quad \cdots \longrightarrow \pi_{k+1}^n \xrightarrow{\Delta} R_k^n \xrightarrow{i_{n+1}^*} R_k^{n+1} \xrightarrow{p_{n+1}^*} \pi_k^n \longrightarrow \cdots.$$

As it is well-known [19, p. 233-4],

$$J(\Delta\alpha) = \pm[\iota_n, \alpha] \quad (\alpha \in \pi_k(S^n)).$$

We also know

$$(5.1) \quad J([\alpha]\beta) = J[\alpha] \circ E^n\beta \quad (\alpha \in \pi_k(S^n), \beta \in \pi_l(S^k)).$$

Although the following is well-known, we show

**Lemma 5.1** *Assume that elements  $\alpha \in \pi_h(SO(n))$ ,  $\beta \in \pi_l(S^h)$  and  $\gamma \in \pi_{k-1}(S^l)$  satisfy the conditions  $\alpha\beta = 0$  and  $\beta\gamma = 0$ . Then the Toda bracket  $\{J\alpha, E^n\beta, E^n\gamma\}_n$  is well-defined and*

$$J\{\alpha, \beta, \gamma\} \subset (-1)^n \{J\alpha, E^n\beta, E^n\gamma\}_n.$$

**Proof.** We recall [17, (11.2)] that the  $J$ -homomorphism is defined by the composition

$$J = G_{n*} \circ E^n : \pi_k(SO(n)) \rightarrow \pi_{k+n}(E^n SO(n)) \rightarrow \pi_{k+n}(S^n).$$

Here  $G_n : E^n SO(n) \rightarrow S^n$  is the Hopf construction obtained from the action of  $SO(n)$  as the rotations of  $S^{n-1}$ . Since  $J\alpha \circ E^n\beta = G_{n*} \circ E^n\alpha \circ E^n\beta = 0$ , the Toda bracket  $\{J\alpha, E^n\beta, E^n\gamma\}_n$  is well-defined. By [17, Propositions 1.2-3], we have

$$\begin{aligned} J\{\alpha, \beta, \gamma\} &= G_{n*} \circ E^n \{ \alpha, \beta, \gamma \} \\ &\subset G_{n*} \circ (-1)^n \{ E^n \alpha, E^n \beta, E^n \gamma \}_n \\ &\subset (-1)^n \{ G_{n*} \circ E^n \alpha, E^n \beta, E^n \gamma \}_n \\ &= (-1)^n \{ J\alpha, E^n \beta, E^n \gamma \}_n. \end{aligned}$$

□

By (5.1), we have

$$(5.2) \quad J(i_{m,n*}\gamma) = (-1)^{n-m} E^{n-m} J\gamma \quad (\gamma \in \pi_k(SO(m))).$$

Since  $\Delta\iota_7 \in \pi_6(SO(7)) = 0$  [7, Table, p. 162], there exists a lift  $[\iota_7] \in R_7^8$  of  $\iota_7$ . By [4, (4.1)], we have

$$(5.3) \quad \Delta\sigma_9 = [\iota_7]_9(\bar{\nu}_7 + \varepsilon_7).$$

By (2.14), we obtain  $\Delta(\sigma_9^2) = [\iota_7]_9(\bar{\nu}_7 + \varepsilon_7)\sigma_{15} = 0$  and so, there exists a lift  $[\sigma_9^2] \in R_{23}^{10}$  of  $\sigma_9^2$ .

We show

**Lemma 5.2**  $\{p_{n+1}, i_{n+1}, \Delta\iota_n\} \ni \iota_n \pmod{2\iota_n}$ .

**Proof.** Let  $P^n$  be the real  $n$  dimensional projective space,  $\gamma_n : S^n \rightarrow P^n$  the projection. Let  $j_n : P^{n-1} \rightarrow SO(n)$  be the canonical inclusion. We know

$$\Delta\iota_n = j_n \gamma_{n-1}, \quad i_n \circ j_n = j_{n+1} \circ i'_n, \quad p_n \circ j_n = p'_{n-1},$$

where  $i'_n : P^{n-1} \rightarrow P^n$  is the inclusion and  $p'_n : P^n \rightarrow S^n$  the collapsing map. We have

$$\begin{aligned} \{p_{n+1}, i_{n+1}, \Delta\iota_n\} &= \{p_{n+1}, i_{n+1}, j_n \gamma_{n-1}\} \\ &\subset \{p_{n+1}, i_n \circ j_n, \gamma_{n-1}\} \\ &= \{p_{n+1}, j_{n+1} \circ i'_n, \gamma_{n-1}\} \\ &\supset \{p_{n+1} \circ j_{n+1}, i'_n, \gamma_{n-1}\} \\ &= \{p'_n, i'_n, \gamma_{n-1}\} \\ &\ni \iota_n \pmod{p_{n+1} \circ \pi_n(SO(n+1)) + [EP^{n-1}, S^n] \circ E\gamma_{n-1}}. \end{aligned}$$

Since  $p_{n+1} \circ \pi_n(SO(n+1)) = \{1 + (-1)^{n-1} \iota_n\}$  and  $[EP^{n-1}, S^n] \circ E\gamma_{n-1} = \{E(p'_{n-1} \circ \gamma_{n-1})\} = \{1 + (-1)^n \iota_n\}$ , we obtain

$$p_{n+1} \circ \pi_n(SO(n+1)) + [EP^{n-1}, S^n] \circ E\gamma_{n-1} = \{2\iota_n\}.$$

This completes the proof.  $\square$

Let us recall the element  $\psi_{10} \in \pi_{33}^{10}$  [9, (4.27)]. We show

**Lemma 5.3**  $J[\iota_7] = \sigma_8$  and  $J[\sigma_9^2] = \psi_{10}$ .

**Proof.** The first is just [4, (2.2)]. Since  $[\iota_7]_{10}(\bar{\nu}_7 + \varepsilon_7) = i_{10} \circ \Delta\sigma_9 = 0$  (5.3), we can define the Toda bracket

$$\{[\iota_7]_{10}, \bar{\nu}_7 + \varepsilon_7, \sigma_{15}\} \subset R_{23}^{10}.$$

By Lemma 5.2 and (5.3), we have

$$\begin{aligned} p_{10} \circ \{[\iota_7]_{10}, \bar{\nu}_7 + \varepsilon_7, \sigma_{15}\} &= -\{p_{10}, [\iota_7]_{10}, \bar{\nu}_7 + \varepsilon_7\} \circ \sigma_{16} \supset -\{p_{10}, i_{10}, \Delta\iota_9 \circ \sigma_8\} \circ \sigma_{16} \\ &\supset -\{p_{10}, i_{10}, \Delta\iota_9\} \circ \sigma_9^2 \ni \sigma_9^2 \pmod{p_{10} \circ R_{16}^{10} \circ \sigma_{16}} = \{2\sigma_9^2\}. \end{aligned}$$

Hence, we can take

$$[\sigma_9^2] \in \{[\iota_7]_{10}, \bar{\nu}_7 + \varepsilon_7, \sigma_{15}\}.$$

By (5.2), we have

$$\begin{aligned} J[\sigma_9^2] &\in J\{[\iota_7]_{10}, \bar{\nu}_7 + \varepsilon_7, \sigma_{15}\} \\ &\subset \{J[\iota_7]_{10}, \bar{\nu}_{17} + \varepsilon_{17}, \sigma_{25}\}_{10} \\ &\subset \{\sigma_{10}, \bar{\nu}_{17} + \varepsilon_{17}, \sigma_{25}\}_4. \end{aligned}$$

Therefore, by the definition of  $\psi_{10}$ , we have the second. This completes the proof.  $\square$

By use of the result in [7, Table, p. 161] and  $(\mathcal{R}_{8n+7}^{8n+5})$ , we obtain

$$\Delta\nu_{8n+5} \neq 0 \ (n \geq 1).$$

In particular, we show

**Lemma 5.4**  $\Delta\nu_{21} = [\sigma_9^2]_{21}$  and  $[\iota_{21}, \nu_{21}] = \psi_{21}$ .

**Proof.** Since  $J\Delta\nu_{21} = -[\iota_{21}, \nu_{21}]$  and  $J[\sigma_9^2]_{21} = J(i_{10,21*}[\sigma_9^2]) = -E^{11}J[\sigma_9^2] = -\psi_{21}$ , the second leads to the first. By [7, Table, p. 161] and [2], we have

$$R_{23}^{21} \cong \mathbf{Z} \oplus \mathbf{Z}_2, \ R_{23}^{22} \cong \mathbf{Z}.$$

By [9, (4.37)] and Lemma 5.3, we observe that

$$J[\sigma_9^2]_{21} = \psi_{21} \neq 0.$$

Hence, the direct summand  $\mathbf{Z}_2$  in  $R_{23}^{21}$  is generated by  $[\sigma_9^2]_{21}$ . We consider the exact sequence  $(\mathcal{R}_{23}^{21})$ :

$$\pi_{24}^{21} \xrightarrow{\Delta} R_{23}^{21} \xrightarrow{i_{21*}} R_{23}^{22} \xrightarrow{p_{22*}} \pi_{23}^{21}.$$

Since  $[\iota_{21}, \eta_{21}^2] = 4\sigma_{21}^* \neq 0$  [8, Lemma 8.3],  $p_{22*}$  is trivial and  $i_{21*}$  is a split epimorphism. Hence, we obtain the relation.  $\square$

The second result in Lemma 5.4 is an excluded case in [16, Theorem 3.6(9)]. This theorem ensures the following.

**Conjecture 5.5** *There exists a lift  $[\sigma_{16k-7}^2] \in R_{16k+7}^{16k-6}$  of  $\sigma_{16k-7}^2$  such that  $\Delta(\nu_{16k+5}) = [\sigma_{16k-7}^2]_{11}$  for  $k \geq 2$ .*

We show

**Lemma 5.6** (1)  $\nu_5^2 \bar{\sigma}_{11} = \eta_5 \bar{\sigma}'_6$ .

(2)  $2(\sigma_9 \nu_{16}^*) \equiv \nu_9^2 \bar{\sigma}_{15} \pmod{4\sigma_9 \xi_{16}}$ ,  $4(\xi_{12} \sigma_{30}) = \nu_{12}^2 \bar{\sigma}_{18}$  and  $\nu_{17}^2 \bar{\sigma}_{23} = 4(\xi_{17} \sigma_{35}) = [\iota_{17}, \eta_{17}^2 \sigma_{19}] \neq 0$ .

**Proof.** We recall from [13, II-Proposition 2.1(7)] the relation

$$\eta_5 \bar{\sigma}'_6 \equiv \nu_5^2 \bar{\sigma}_{11} \pmod{\eta_5 \bar{\mu}_6 \sigma_{23}}.$$

In the stable range,  $\nu^2 \bar{\sigma} = \eta \circ \langle \nu, \eta, \bar{\sigma} \rangle = \eta \bar{\sigma}'$  and  $\eta \bar{\mu} \sigma \neq 0$ . This leads to (1).

We recall from [13, p. 70 and I-Proposition 5.1(1)] the relations

$$2\phi''' \equiv \nu_7^2 \bar{\sigma}_{13} \pmod{2E^2 \phi''},$$

$$2E^2 \phi'' \equiv 2(\sigma' \xi_{14}) \pmod{2(\sigma'(E\lambda + 2\xi_{14}))}$$

and

$$2E^4 \phi'' \equiv 4(\sigma_9 \xi_{16}) \pmod{4(\sigma_9(E^3 \lambda)) + 8(\sigma_9 \xi_{16})}.$$

We know  $8(\sigma_9\xi_{16}) = 0$  and  $\sigma_9(E^3\lambda) = 2(\sigma_9\nu_{16}^*)$  [13, I-Proposition 3.1(8)],  $2(\sigma_9\nu_{16}^*) = 2E^2\phi'''$  and  $4(\sigma_9\nu_{16}^*) = 0$  [13, I-Proposition 5.1(2);(3)]. This leads to the first of (2).

By using [17, Corollary 12.25] and [13, I-Theorem 1(a), Propositions 3.5(3), 6.3(8);(11)], we have

$$2(\xi_{12}\sigma_{30}) = \sigma_{12}\xi_{19}, [\iota_{12}, \sigma_{12}^2] = \sigma_{12}(\xi_{19} + \nu_{19}^*) \text{ and } [\iota_{17}, \eta_{17}^2\sigma_{19}] = 4(\xi_{17}\sigma_{35}).$$

This leads to the rest of (2) and completes the proof.  $\square$

Denote by  $M^n = S^{n-1} \cup_{2\iota_{n-1}} e^n$  be  $\mathbf{Z}_2$ -Moore space and by  $i_n : S^{n-1} \rightarrow M^n$  the inclusion map. Let  $\bar{\eta}_n \in [M^{n+2}, S^n]$ ,  $\tilde{\eta}_n \in \pi_{n+2}(M^{n+1})$  ( $n \geq 3$ ) be an extension and a coextension of  $\eta_n$ , respectively. Notice that  $\tilde{\eta}_n \in \pi_{n+2}(M^{n+1}) \cong \mathbf{Z}_4$  is a generator. We note that  $2\nu_n = \pm\bar{\eta}_n\tilde{\eta}_{n+1}$  for  $n \geq 5$ . We show

**Lemma 5.7**  $\delta \equiv \eta\eta^*\sigma \pmod{\bar{\mu}\sigma}$  and  $\delta' = \eta\eta^*\sigma$ .

**Proof.** By [13, I-Propositions 3.4(7), 3.5(9)], we have

$$\eta_8\sigma_9\eta_{16}^* \equiv \delta_8 \pmod{\bar{\mu}_8\sigma_{25}}, \eta_8\sigma_9^2\mu_{23}.$$

We know  $\eta_9\sigma_{10}^2\mu_{24} = 0$  (2.23). This leads to the relation

$$(5.4) \quad \eta_9\sigma_{10}\eta_{17}^* \equiv \delta_9 \pmod{\bar{\mu}_9\sigma_{26}}$$

and the first half.

By the relations  $\eta\sigma = \bar{\nu} + \varepsilon$  and  $\varepsilon\omega = \eta\eta^*\sigma$  [9, (6.3)], we have

$$\langle \varepsilon, \sigma, \eta\sigma \rangle \subset \langle \varepsilon, \sigma, \bar{\nu} \rangle + \langle \varepsilon, \sigma, \varepsilon \rangle.$$

By [9, (6.1)], we have

$$\langle \varepsilon, \sigma, \bar{\nu} \rangle = \{\eta\eta^*\sigma\}.$$

We recall the relation [5, Lemma 4.1]:

$$\langle \eta\bar{\eta}, \tilde{\eta}, \nu \rangle = \varepsilon.$$

By the Jacobi identity, we have

$$\begin{aligned} \langle \varepsilon, \sigma, \varepsilon \rangle &= \langle \langle \eta\bar{\eta}, \tilde{\eta}, \nu \rangle, \sigma, \varepsilon \rangle \\ &\equiv \langle \eta\bar{\eta}, \langle \tilde{\eta}, \nu, \sigma \rangle, \varepsilon \rangle + \langle \eta\bar{\eta}, \tilde{\eta}, \langle \nu, \sigma, \varepsilon \rangle \rangle \pmod{\{\varepsilon\omega\} + \eta\bar{\eta} \circ \pi_{23}^s(M^2)}. \end{aligned}$$

We know  $\langle \tilde{\eta}, \nu, \sigma \rangle \subset \pi_{14}^s(M^2) = 0$  and  $\langle \nu, \sigma, \varepsilon \rangle = 0$  (Lemma 2.2). This implies

$$\langle \varepsilon, \sigma, \varepsilon \rangle \ni 0 \pmod{\{\varepsilon\omega\} + \eta\bar{\eta} \circ \pi_{23}^s(M^2)}.$$

Let  $\widetilde{\sigma^2} \in \pi_{16}^s(M^2)$  be a coextension of  $\sigma^2$ . Then, by the definition of  $\eta^*$  [17, p. 153] and [17, (3.9).i)], we have

$$\bar{\eta}\widetilde{\sigma^2} \in \langle \eta, 2\iota, \sigma^2 \rangle \ni \eta^* \pmod{\eta\rho = \mu\sigma}.$$

By easy calculations making use of the cofibration  $S^1 \rightarrow M^2 \rightarrow S^2$ , we obtain

$$\pi_{23}^s(M^2) = \{\tilde{\eta}\bar{\kappa}, \widetilde{\sigma^2\sigma}, i\nu\bar{\sigma}\} \cong \mathbf{Z}_4 \oplus (\mathbf{Z}_2)^2 \quad \text{and} \quad 2\tilde{\eta}\bar{\kappa} = i\eta^2\bar{\kappa} \quad (i = E^\infty i_2).$$

By the relations  $2\nu = \pm\tilde{\eta}\bar{\eta}$ ,  $\eta\sigma^2 = 0$  and  $\bar{\eta}i\nu = \eta\nu = 0$ , we have

$$\eta\bar{\eta} \circ \pi_{23}^s(M^2) = \eta \circ \{2\nu\bar{\kappa}, \eta^*\sigma\} = \{\eta\eta^*\sigma\}.$$

This implies

$$\delta' \equiv 0 \pmod{\eta\eta^*\sigma}.$$

Hence, the second half follows from the first half and Lemma 4.5. This completes the proof.  $\square$

Now we show

**Theorem 5.8**  $\delta'_9 = \eta_9\sigma_{10}\eta_{17}^*$  and  $\bar{\sigma}''_{19} + \delta'_{19} = [\iota_{19}, \nu_{19}^2]$ .

**Proof.** By the similar proof to [13, II-Proposition 2.1(10)], we obtain

$$\eta_9\psi_{10} \equiv \bar{\sigma}''_9 + \delta'_9 \pmod{\bar{\mu}_9\sigma_{26}, \sigma_9^2\eta_{23}\mu_{24}}.$$

Since  $\eta_{20}\psi_{21} = [\eta_{20}, \eta_{20}\nu_{21}] = 0$  by Lemma 5.4, we obtain

$$(5.5) \quad \bar{\sigma}''_{20} \equiv \delta'_{20} \pmod{\bar{\mu}_{20}\sigma_{37}}.$$

By the fact that  $\pi_{44}^{20} \cong \pi_{24}^s$  and Lemma 5.7, we have

$$(5.6) \quad \eta_{20}\eta_{21}^*\sigma_{37} = \eta_{20}\sigma_{21}\eta_{28}^* \quad \text{and} \quad \delta'_{20} = \eta_{20}\eta_{21}^*\sigma_{37}.$$

From (5.5) and (5.6), we have  $0 = \bar{\sigma}''_{20} + \eta_{20}\eta_{21}^*\sigma_{37} + a\bar{\mu}_{20}\sigma_{37}$  for  $a = 0$  or  $a = 1$ , and  $\bar{\sigma}'' = \eta\eta^*\sigma + a\bar{\mu}\sigma$ . By Lemma 5.6(2), we have  $\nu^2\bar{\sigma} = 0$ . Hence, by [17, Lemma 5.12 and 14.1.i)] and Lemma 4.4, we have

$$0 = \nu^2\bar{\sigma} = \langle \eta, \nu, \eta \rangle \circ \bar{\sigma} = \eta \circ \langle \nu, \eta, \bar{\sigma} \rangle = \eta(\eta\eta^*\sigma + a\bar{\mu}\sigma) = 4\nu^*\sigma + a\eta\bar{\mu}\sigma = a\eta\bar{\mu}\sigma.$$

This induces  $a = 0$  and the second relation.

The third one follows from [9, (5.38-40)] and Lemma 4.5.

By Lemma 4.5 and (5.4), we have

$$\delta'_9 \equiv \eta_9\sigma_{10}\eta_{17}^* \pmod{\bar{\mu}_9\sigma_{16}}.$$

So, in the stable range, we have

$$\delta' \equiv \eta\sigma\eta^* \pmod{\bar{\mu}\sigma}.$$

This and Lemma 5.7 lead to the first and completes the proof.  $\square$

Since  $\varepsilon_9 \circ E^4\pi_{29}^{13} = \{\varepsilon_9\sigma_{17}\mu_{24}\} = 0$  and

$$\pi_{25}^9 \circ \eta_{25}\sigma_{26} = \{\sigma_9\nu_{16}^3, \sigma_9\mu_{16}, \sigma_9\eta_{15}\varepsilon_{16}, \mu_9\sigma_{18}\} \circ \eta_{25}\sigma_{26} = 0,$$

we obtain  $\text{Ind}\{\varepsilon_9, \sigma_{17}, \eta_{24}\sigma_{25}\}_4 = 0$ . Lemmas 4.4, 5.7 and Theorem 5.8 imply Theorem 1.1.

By [1, Proposition], [11, Example 2.3(3)] and Lemma 5.4,

$$\sigma_{21}\omega_{28} - \omega_{21}\sigma_{37} = [\iota_{21}, \nu_{21}] = \eta_{21}\sigma_{22}^* = \psi_{21}.$$

We recall from [17, p. 153] that the definition of  $\eta^{*'} \in \pi_{31}^{15}$  is

$$(5.7) \quad \eta^{*'} \in \{\sigma_{15}, 4\sigma_{22}, \eta_{29}\}_1.$$

We show

**Lemma 5.9**  $E\eta^{*'} \circ \sigma_{32} = [\iota_{16}, \eta_{16}\sigma_{17}]$  and  $\eta_{16}\sigma_{17}^* = \sigma_{16}\omega_{23} + \omega_{16}\sigma_{32} + [\iota_{16}, \eta_{16}\sigma_{17}]$ .

**Proof.** By (5.7),  $2\sigma_{16}^2 = 0$ ,  $\pi_{31}^{16} = \{\rho_{16}, \bar{\varepsilon}_{16}, [\iota_{16}, \iota_{16}]\}$ , and  $\rho_{16}\eta_{31} = \sigma_{16}\mu_{23}$  [17, Proposition 12.20.i)], we obtain

$$E\eta^{*'} \in \{\sigma_{16}, 4\sigma_{23}, \eta_{30}\} \supset \{0, 2\iota_{30}, \eta_{30}\} = \pi_{31}^{16} \circ \eta_{31} = \{\sigma_{16}\mu_{23}, [\iota_{16}, \eta_{16}]\}.$$

Since  $\sigma_{16}\mu_{23}\sigma_{32} = \sigma_{16}^2\mu_{30} = 0$  from [9, (2.3)], we have

$$E\eta^{*'} \circ \sigma_{32} \in \{[\iota_{16}, \eta_{16}\sigma_{17}]\}.$$

This and the relation  $[\iota_{16}, \eta_{16}] \equiv E\eta^{*'} \pmod{E^2\pi_{30}^{14}}$  [17, p. 160] lead to the first half.

By the relation  $\eta_{15}\sigma_{16}^* = \eta^{*'}\sigma_{31} + \sigma_{15}\omega_{22} + \omega_{15}\sigma_{31}$  [13, III-Proposition 2.5(5)] and the first half leads to the second half and the proof is complete.  $\square$

Finally we show

**Proposition 5.10**  $\eta_{16}^*\bar{\nu}_{32} \equiv \nu_{16}^*\nu_{34}^2 \pmod{[\iota_{16}, \nu_{16}^3]}$  and  $\nu_{19}^*\nu_{37}^2 = \eta_{19}^*\bar{\nu}_{35} = \omega_{19}\bar{\nu}_{35} = [\iota_{19}, \nu_{19}^2]$ .

**Proof.** The last equality is just [9, (5.34)].

We have  $\eta_{16}^*\nu_{32} = 0$ , because we see that

$$\begin{aligned} \eta_{16}^*\nu_{32} &\in \{\sigma_{16}, 2\sigma_{23}, \eta_{30}\} \circ \nu_{32} = -\sigma_{16} \circ \{2\sigma_{23}, \eta_{30}, \nu_{31}\} \\ &\supset \sigma_{16}^2 \circ \{2\iota_{30}, \eta_{30}, \nu_{31}\} = 0 \pmod{\sigma_{16} \circ \pi_{32}^{23} \circ \nu_{32} = 0}. \end{aligned}$$

By using [17, Lemma 6.2],  $\pi_{37}^{16} = \{\eta_{16}\bar{\kappa}_{17}, \sigma_{16}^3, (E^3\lambda)\nu_{34}, \nu_{16}^*\nu_{34}\}$  [8, Theorem A],  $\eta_{16}\bar{\kappa}_{17}\nu_{37} = \bar{\kappa}_{16}\eta_{36}\nu_{37} = 0$ ,  $\sigma_{30}\nu_{37} = 0$  and  $(E^3\lambda)\nu_{34} = [\iota_{16}, \nu_{16}^2]$  [8, (7.10)], we obtain

$$\begin{aligned} \eta_{16}^*\bar{\nu}_{32} &\in \eta_{16}^* \circ \{\nu_{32}, \eta_{35}, \nu_{36}\} = \{\eta_{16}^*, \nu_{32}, \eta_{35}\} \circ \nu_{37} \subset \pi_{37}^{16} \circ \nu_{37} \\ &= \{\eta_{16}\bar{\kappa}_{17}\nu_{37}, \sigma_{16}^3\nu_{37}, (E^3\lambda)\nu_{34}^2, \nu_{16}^*\nu_{34}^2\} = \{[\iota_{16}, \nu_{16}^2], \nu_{16}^*\nu_{34}^2\}. \end{aligned}$$

By [17, Lemma 12.14] and (2.6), we have  $H(\eta_{16}^*\bar{\nu}_{32}) = \nu_{31}^3 = H(\nu_{16}^*\nu_{34}^2)$ . Hence, by the EHP-sequence, we have the first assertion and  $\eta_{19}^*\bar{\nu}_{35} = \nu_{19}^*\nu_{37}^2$ . By (2.16) and (2.17), we have

$$[\iota_{19}, \nu_{19}] = \nu_{19}^*\nu_{37} + \sigma_{19}^3.$$

Therefore we obtain  $[\iota_{19}, \nu_{19}^2] = \nu_{19}^*\nu_{37}^2$ . This leads to the second assertion and completes the proof.  $\square$

## 6 Proof of Theorem 1.3

We recall the element [8, p. 187]

$$(6.1) \quad \sigma_{16}^* \in \{\sigma_{16}, 2\sigma_{23}, \sigma_{30}\}_1.$$

We also recall [13, II-Proposition 2.1(4)]:

$$(6.2) \quad 4\sigma_{14}\rho_{21} = 2\sigma_{15}\rho_{22} = \sigma_{17}\rho_{24} = 0.$$

We recall from [13, I-Proposition 3.4(6)] the relation

$$\{\eta_{15}, 2\sigma_{16}, \sigma_{23}\} \ni \omega_{15} + \eta^{*'} \pmod{\eta_{15} \circ \pi_{31}^{16} + \pi_{24}^{15} \circ \sigma_{24}}.$$

Since  $\pi_{31}^{16} = \{\rho_{16}, \bar{\varepsilon}_{16}, [\iota_{16}, \iota_{16}]\}$  [17, Theorem 10.10],  $\eta_{15}\rho_{16} = \mu_{15}\sigma_{24} = \sigma_{15}\mu_{22}$  [17, Proposition 12.20.i)],  $\eta_{15}\bar{\varepsilon}_{16} = \nu_{15}\sigma_{18}\nu_{25}^2 = 0$  ([17, Lemma 12.10], (2.10)) and  $\eta_{15} \circ [\iota_{16}, \iota_{16}] = [\iota_{15}, \eta_{15}^2] = 0$  [3, Theorem], we have  $\eta_{15} \circ \pi_{31}^{16} = \{\sigma_{15}\mu_{22}\}$ . Since  $\pi_{29}^{15} = \{\nu_{15}^3, \mu_{15}, \eta_{15}\varepsilon_{16}\}$  [17, Theorem 7.2], (2.10) and (2.14), we have  $\pi_{24}^{15} \circ \sigma_{24} = \{\sigma_{15}\mu_{22}\}$ . Hence, we obtain

$$(6.3) \quad \{\eta_{15}, 2\sigma_{16}, \sigma_{23}\} \ni \omega_{15} + \eta^{*'} \pmod{\sigma_{15}\mu_{22}}.$$

We show

**Lemma 6.1**  $\bar{\nu}_7\eta^{*'} = 0$ .

**Proof.** Since  $E : \pi_{31}^7 \rightarrow \pi_{32}^8$  is a monomorphism [9, Theorem 1.1(a)], it suffices to show  $\bar{\nu}_8(E\eta^{*'}) = 0$ . By (5.7) and (2.14), we have

$$\bar{\nu}_8(E\eta^{*'}) \in \bar{\nu}_8 \circ -\{\sigma_{16}, 4\sigma_{23}, \eta_{30}\} = \{\bar{\nu}_8, \sigma_{16}, 4\sigma_{23}\} \circ \eta_{31}.$$

We observe

$$\{\bar{\nu}_8, \sigma_{16}, 4\sigma_{23}\} \subset \{\bar{\nu}_8, 2\sigma_{16}, 2\sigma_{23}\} \supset \{0, \sigma_{16}, 2\sigma_{23}\} \ni 0 \pmod{\bar{\nu}_8 \circ \pi_{31}^{16} + \pi_{24}^8 \circ 2\sigma_{24}}.$$

By relations  $\bar{\nu}_8\rho_{16} = 0$  [13, I-Proposition 3.1(4)],  $\bar{\varepsilon}_{16} = \eta_{16}\kappa_{17}$  (2.18),  $\bar{\nu}_8\eta_{16} = \nu_8^3$  (2.6),  $\nu_{11}^2\kappa_{17} = 4\bar{\kappa}_{11}$  [10, Theorem 15.4] and  $\bar{\nu}_8 \circ [\iota_{16}, \iota_{16}] = [\iota_8, \iota_8] \circ \bar{\nu}_{15}^2 = 0$  [15, Proposition 2.8(2)], we have  $\bar{\nu}_8 \circ \pi_{31}^{16} = \{4\nu_8\bar{\kappa}_{11}\}$ . Since  $2\pi_{24}^8 = 0$ , we have  $\pi_{24}^8 \circ 2\sigma_{24} = 0$ . Hence, we obtain  $\bar{\nu}_8(E\eta^{*'}) \subset \{4\nu_8\bar{\kappa}_{11}\} \circ \eta_{31} = 0$ . This completes the proof.  $\square$

Next, we show

**Lemma 6.2**  $\bar{\nu}_6\omega_{14} \equiv P(\lambda + \xi_{13}) \circ \eta_{29} \pmod{\nu_6\sigma_9\kappa_{16}, 4\bar{\zeta}_6'}$ .

**Proof.** Apply [8, Corollary 5.10] to the case  $\alpha = \bar{\nu}_6$ ,  $\beta = \omega_{14}$ ,  $n = 5$ ,  $p = 13$ ,  $i = 29$ . Then, we have

$$H(\bar{\nu}_6\omega_{14}) \equiv \nu_{11}\omega_{14} + \bar{\nu}_{11}^2\nu_{27} \pmod{G},$$



where

$$G = \sum_{k=3}^6 f_{k*} \pi_{30}^{5k+1} + \text{Ker}\{E : \pi_{29}^{10} \rightarrow \pi_{30}^{11}\}.$$

We know  $\bar{\nu}_{19}\nu_{27} = 0$  (2.12) and  $\text{Ker}\{E : \pi_{29}^{10} \rightarrow \pi_{30}^{11}\} = 0$ . We also have  $f_3 \in \pi_{16}^{11} = 0$  and  $\pi_{30}^{5k+1} = 0$  for  $k = 5, 6$ . Hence, we obtain

$$H(\bar{\nu}_6\omega_{14}) \equiv \nu_{11}\omega_{14} \pmod{f_{4*}\pi_{30}^{21}}.$$

Since  $\pi_{21}^{11} \circ \pi_{30}^{21} = \{4\bar{\zeta}_{11}\}$ , we get that

$$H(\bar{\nu}_6\omega_{14}) \equiv \nu_{11}\omega_{14} \pmod{4\bar{\zeta}_{11}}.$$

By relations (2.28), Lemma 3.5 and  $H(\bar{\zeta}_6') \equiv \bar{\zeta}_{11} \pmod{2\bar{\zeta}_{11}}$  [9, (3.8)], we have

$$\begin{aligned} H(\bar{\nu}_6\omega_{14}) &\equiv \nu_{11}\omega_{14} = \lambda'\eta_{29} + \xi'\eta_{29} \\ &= H(P(\lambda\eta_{31})) + H(P(\xi_{13}\eta_{31})) \pmod{H(4\bar{\zeta}_6')}. \end{aligned}$$

Therefore, we see that

$$\bar{\nu}_6\omega_{14} \equiv P(\lambda\eta_{31}) + P(\xi_{13}\eta_{31}) + 4a\bar{\zeta}_6' \pmod{E\pi_{29}^5}$$

for  $a = 0$  or  $a = 1$ . By (2.9) and  $2\kappa_{19} = 0$ , we have  $\nu_9\sigma_{12}\kappa_{19} = (E^2\sigma')\nu_{16}\kappa_{19} = 2\sigma_9\nu_{16}\kappa_{19} = 0$ . We know  $E\pi_{29}^5 = \{\delta_6, \bar{\mu}_6\sigma_{23}, \nu_6\sigma_9\kappa_{16}\} \cong (\mathbf{Z}_2)^3$ ,  $E^6\pi_{29}^5 = \{\delta_{11}, \bar{\mu}_{11}\sigma_{28}\} \cong (\mathbf{Z}_2)^2$ ,  $\bar{\nu}_{11}\omega_{19} = 0$  (4.1). Hence, we conclude that

$$\bar{\nu}_6\omega_{14} \equiv P(\lambda\eta_{31}) + P(\xi_{13}\eta_{31}) = P(\lambda + \xi_{13})\eta_{29} \pmod{\nu_6\sigma_9\kappa_{16}, 4\bar{\zeta}_6'}.$$

This completes the proof.  $\square$

By (2.6), (2.14), (6.3) and Lemma 6.1, we have

$$\begin{aligned} \bar{\nu}_7\omega_{15} &\in \bar{\nu}_7 \circ \{\eta_{15}, 2\sigma_{16}, \sigma_{23}\} \subset \{\nu_7^3, 2\sigma_{16}, \sigma_{23}\} \\ &\supset \nu_7 \circ \{\nu_{10}^2, 2\sigma_{16}, \sigma_{23}\} \pmod{\nu_7^3 \circ \pi_{31}^{16} + \pi_{24}^7 \circ \sigma_{24}}. \end{aligned}$$

By relations (2.31), (2.18) and  $[\iota_{13}, \nu_{13}] = 0$ , we have

$$\nu_{13} \circ \pi_{31}^{16} = \nu_{13} \circ \{\rho_{16}, \bar{\varepsilon}_{16}, [\iota_{16}, \iota_{16}]\} = 0 \quad \text{and} \quad \nu_7^3 \circ \pi_{31}^{16} = 0.$$

We recall  $\pi_{24}^7 = \{\sigma'\eta_{14}\mu_{15}, \nu_7\kappa_{10}, \bar{\mu}_7, \eta_7\mu_8\sigma_{17}\}$ . By relations (2.8),  $\bar{\nu}_6\mu_{14} = 0$  (2.20) and  $\sigma'\varepsilon_{14}\mu_{22} = \bar{\zeta}_7'$  [9, (5.10)], we obtain

$$\sigma'\eta_{14}\mu_{15}\sigma_{24} = \sigma'\eta_{14}\sigma_{15}\mu_{22} = \sigma'(\bar{\nu}_{14} + \varepsilon_{14})\mu_{22} = \bar{\zeta}_7'.$$

Hence, by relations  $\nu_7\kappa_{10}\sigma_{24} = 0$  (3.3) and  $\eta_7\mu_8\sigma_{17}^2 = 0$  (4.2), we have  $\pi_{24}^7 \circ \sigma_{24} = \{\bar{\zeta}_7', \bar{\mu}_7\sigma_{24}\}$  and

$$\bar{\nu}_7\omega_{15} \in \nu_7 \circ \{\nu_{10}^2, 2\sigma_{16}, \sigma_{23}\} \pmod{\bar{\zeta}_7', \bar{\mu}_7\sigma_{24}}.$$

By relations (3.15), (3.1), (2.17),  $2\kappa_{17} = 0$ ,  $2\sigma_{17}^2 = 0$  and (3.3), for some odd integer  $x$ , we obtain

$$\begin{aligned} \{\nu_{10}^2, 2\sigma_{16}, \sigma_{23}\} &\supset \nu_{10} \circ \{\nu_{13}, 2\sigma_{16}, \sigma_{23}\} = \nu_{10}\xi_{13} + x\nu_{10}\lambda + 2x\nu_{10}\xi_{13} \\ &= \sigma_{10}^3 + \sigma_{10}\kappa_{17} \bmod \nu_{10}^2 \circ \pi_{31}^{16} + \pi_{24}^{10} \circ \sigma_{24} = \{\sigma_{10}^3\}. \end{aligned}$$

By the relations (2.9),  $2\sigma_{17}^2 = 0$  and (2.10), we have  $\nu_7\sigma_{10}^3 = \sigma'\nu_{14}\sigma_{17}^2 = 0$ . Hence, we obtain

$$\bar{\nu}_7\omega_{15} \equiv \nu_7\sigma_{10}\kappa_{17} \bmod \bar{\zeta}_7', \bar{\mu}_7\sigma_{24}.$$

The first 3 elements become trivial and the last survives in the stable range. This induces the relation

$$\bar{\nu}_7\omega_{15} \equiv \nu_7\sigma_{10}\kappa_{17} \bmod \bar{\zeta}_7'.$$

On the other hand, by Lemma 6.2, we have

$$\bar{\nu}_7\omega_{15} \equiv 0 \bmod \nu_7\sigma_{10}\kappa_{17}.$$

Hence we obtain the equation  $\bar{\nu}_7\omega_{15} = \nu_7\sigma_{10}\kappa_{17}$ . This completes the proof of Theorem 1.3.

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